# *Displacements Depending on One, Two, ...., k Parameters in a Space of n Dimensions.*

By N. J. Hatzidakis.

1. Professor Craig has recently considered (this Journal, vol. XX, No. 2, April, 1898) the displacements in a space of four dimensions and generalized the theory of M. Darboux. In the present short paper I shall examine the general case of displacements in a space of $n$ dimensions.

2. In the ordinary space of three dimensions, the curves and the surfaces correspond to the displacements depending on one or two parameters. No relations exist between the kinematic elements of the curves; but for the surfaces six relations exist between the ten kinematic elements (the two others, $\zeta$ and $\zeta_1$, are $= 0$). These relations are the *fundamental kinematic equations* of the surfaces. This agrees perfectly with the usual analytical theory of the surfaces, which gives *three* fundamental equations between the *six* elements of Gauss, $E$, $F$, $G$, $D$, $D'$, $D''$.* It suffices to remember the relations existing between the kinematic and the analytical elements.†

3. In the general case of a linear space of $n$ dimensions, $n-1$ kinds of displacements are to be considered, viz., those which depend on one, two, ...., $n-1$ parameters. To the *one-parametric* displacements correspond the curves of this space; to the *two-parametric*, the surfaces; to the *three-parametric*, the *hypersurfaces*, or *manifoldnesses* of three dimensions, etc.; to the *k-parametric* displacements correspond the *manifoldnesses* (Variétés, Gebilde) of $k$ dimensions. No

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*Darboux, "Surfaces," vol. III, pp. 247–8, or Crelle's Journal, vol. LXXXVIII, p. 69 (1880).

†Darboux, "Surfaces," vol. II, pp. 376 and 379.

relations exist for the kinematic elements of the curves in this general space; for all the other manifoldnesses, however, there exist many relations which we shall now find.

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4. In the $n$-dimensional space, let $OX_1 X_2 \ldots\ldots X_n$ be a system of fixed axes and $ox_1 x_2 \ldots\ldots x_n$ a movable system. Suppose both systems orthogonal.

The $n$ equations

$$x_1 = 0, \quad x_2 = 0, \ldots\ldots, x_n = 0,$$

(each separately) give $n$ $(n-1)$-dimensional linear spaces, viz., the $(n-1)$-dimensional coordinate spaces in the moving system. Further, the $\frac{n(n-1)}{2}$ systems of equations

$$\begin{array}{llll} x_1 = 0, x_2 = 0; & x_1 = 0, x_3 = 0; & \ldots\ldots; & x_1 \quad = 0, x_n = 0; \\ & x_2 = 0, x_3 = 0; & \ldots\ldots; & x_2 \quad = 0, x_n = 0; \\ & & \ldots\ldots & \ldots\ldots\ldots\ldots\ldots\ldots \\ & & & x_{n-1} = 0, x_n = 0 \end{array}$$

(each separately), give $\frac{n(n-1)}{2}$ $(n-2)$-dimensional coordinate spaces in the moving system. Generally, we find $\frac{n(n-1)\ldots\ldots(n-k+1)}{k!}$ coordinate spaces of the $(n-k)^{\text{th}}$ dimension. For $k = n-1$, we have the $n$ axes. But among all these kinds of spaces, those of the $(n-2)^{\text{th}}$ dimension correspond, for the rotations, to the axes of the three-dimensional space. I shall, consequently, call the figure formed by these spaces the *moving polyhedron* (par excellence).

5. Let, now, the following be the *schema* of the cosines of the movable axes with the fixed

$$\begin{array}{c|ccccc} & x_1 & x_2 & \ldots\ldots & x_{n-1} & x_n \\ \hline X_1 & \alpha_1 & \alpha_2 & \ldots\ldots & \alpha_{n-1} & \alpha_n \\ X_2 & \beta_1 & \beta_2 & \ldots\ldots & \beta_{n-1} & \beta_n \\ \ldots\ldots & \ldots & \ldots & \ldots\ldots & \ldots & \ldots \\ X_{n-1} & \mu_1 & \mu_2 & \ldots\ldots & \mu_{n-1} & \mu_n \\ X_n & \nu_1 & \nu_2 & \ldots\ldots & \nu_{n-1} & \nu_n \end{array} \tag{1}$$

The determinant of these cosines being orthogonal, we have the relations

$$\left.\begin{array}{rrrr}
\sum_{i=1}^{n}\alpha_i^2=1, & \sum_{i=1}^{n}\beta_i^2=1, & \sum_{i=1}^{n}\gamma_i^2=1, \;\ldots\ldots, & \sum_{i=1}^{n}\nu_i^2=1, \\
& \sum_{i=1}^{n}\alpha_i\beta_i=0, & \sum_{i=1}^{n}\alpha_i\gamma_i=0, \;\ldots\ldots, & \sum_{i=1}^{n}\alpha_i\nu_i=0, \\
& & \sum_{i=1}^{n}\beta_i\gamma_i=0, \;\ldots\ldots, & \sum_{i=1}^{n}\beta_i\nu_i=0, \\
& & & \ldots\ldots\ldots\ldots\ldots\ldots \\
& & & \sum_{i=1}^{n}\mu_i\nu_i=0,
\end{array}\right\} \qquad (2)$$

or the equivalent set

$$\left.\begin{array}{rrrr}
\sum\alpha_1^2=1, & \sum\alpha_2^2=1, & \sum\alpha_3^2=1, \;\ldots\ldots, & \sum\alpha_n^2=1, \\
& \sum\alpha_1\alpha_2=0, & \sum\alpha_1\alpha_3=0, \;\ldots\ldots, & \sum\alpha_1\alpha_n=0, \\
& & \sum\alpha_2\alpha_3=0, \;\ldots\ldots, & \sum\alpha_2\alpha_n=0, \\
& & & \ldots\ldots\ldots\ldots\ldots\ldots \\
& & & \sum\alpha_{n-1}\alpha_n=0,
\end{array}\right\} \qquad (2')$$

where the $\Sigma$'s are extended over all the cosines having the same *index*.

6. In the general motion of a system consisting of a rotation and a translation, we consider at first the case of the rotation, supposing that the moving axes have a fixed origin, namely, that of the $OX_1X_2\ldots\ldots X_n$. We have then

$$V_{x_i}=\alpha_iV_{X_1}+\beta_iV_{X_2}+\ldots\ldots+\nu_iV_{X_n}+\frac{dx_i}{dt},$$
$$(i=1,\,2,\,\ldots\ldots,\,n)$$

$$\left.\begin{array}{ll}
\text{or} & V_{x_i}=\alpha_i\sum_{j=1}^{n}x_j\frac{d\alpha_j}{dt}+\beta_i\sum_{j=1}^{n}x_j\frac{d\beta_j}{dt}+\ldots\ldots+\nu_i\sum_{j=1}^{n}x_j\frac{d\nu_j}{dt}+\frac{dx_i}{dt}, \\
\text{viz.,} & V_x=x_1\sum\alpha_i\frac{d\alpha_1}{dt}+x_2\sum\alpha_i\frac{d\alpha_2}{dt}+\ldots\ldots+x_n\sum\alpha_i\frac{d\alpha_n}{dt}+\frac{dx_i}{dt},
\end{array}\right\} \qquad (3)$$

the $\Sigma$'s extended over all the cosines having the same index. $V_{x_i}$ denotes the components of the absolute velocity of a point $x_i$ relative to the moving axes.*

If we now write for the sake of brevity

$$\left.\begin{array}{r}\sum \alpha_2 \frac{d\alpha_1}{dt} = p_{12}, \sum \alpha_3 \frac{d\alpha_1}{dt} = p_{13}, \ldots, \sum \alpha_n \frac{d\alpha_1}{dt} = p_{1n}, \\ \sum \alpha_3 \frac{d\alpha_2}{dt} = p_{23}, \ldots, \sum \alpha_n \frac{d\alpha_2}{dt} = p_{2n}, \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ \sum \alpha_n \frac{d\alpha_{n-1}}{dt} = p_{n-1n},\end{array}\right\} \quad (4)$$

equations (3) take the form

$$\left.\begin{array}{l}V_{x_1} = \frac{dx_1}{dt} \qquad\qquad - x_2 p_{12} - x_3 p_{13} - \ldots\ldots\ldots - x_n p_{1n}, \\ V_{x_2} = \frac{dx_2}{dt} + x_1 p_{12} \qquad\qquad - x_3 p_{23} - \ldots\ldots\ldots - x_n p_{2n}, \\ V_{x_3} = \frac{dx_3}{dt} + x_1 p_{13} + x_2 p_{23} \qquad\qquad - x_4 p_{34} - \ldots\ldots - x_n p_{3n}, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ V_{x_{n-1}} = \frac{dx_{n-1}}{dt} + x_1 p_{1\,n-1} + x_2 p_{2\,n-1} + \ldots\ldots + x_{n-2} p_{n-2\,n-1} - x_n p_{n-1n}, \\ V_{x_n} = \frac{dx_n}{dt} + x_1 p_{1n} + x_2 p_{2n} + \ldots\ldots\ldots + x_{n-1} p_{n-1n}.\end{array}\right\} \quad (5$$

7. It is now very easy to find the equations for the cosines. It suffices, of course, to consider the points having coordinates relatively to the axes $OX_1 X_2 \ldots . X_n : 1, 0, \ldots, 0;\ 0, 1, 0, \ldots, 0$, etc., $0, 0, \ldots, 0, 1$.† We find so the following $n^2$ equations

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* Cf. Appell, "Mécanique," vol. I, pp. 62–3.

† Cf. Darboux, "Surfaces," vol. I, p. 4.

$$\left.\begin{aligned}
\frac{d\alpha_1}{dt} &= \sum_{\lambda=2}^{n} \alpha_\lambda p_{1\lambda}, \\
\frac{d\alpha_2}{dt} &= \sum_{\lambda=3}^{n} \alpha_\lambda p_{2\lambda} - \alpha_1 p_{12}, \\
\frac{d\alpha_3}{dt} &= \sum_{\lambda=4}^{n} \alpha_\lambda p_{3\lambda} - \sum_{\tau=1}^{2} \alpha_\tau p_{\tau 3}, \\
&\cdots\cdots\cdots\cdots\cdots\cdots \\
\frac{d\alpha_i}{dt} &= \sum_{\lambda=i+1}^{n} \alpha_\lambda p_{i\lambda} - \sum_{\tau=1}^{i-1} \alpha_\tau p_{\tau i}, \\
&\cdots\cdots\cdots\cdots\cdots\cdots \\
\frac{d\alpha_n}{dt} &= \qquad\qquad - \sum_{\tau=1}^{n-1} \alpha_\tau p_{\tau n},
\end{aligned}\right\} \tag{6}$$

and $n - 1$ other sets of equations for the other cosines $\beta, \gamma, \ldots, \mu, \nu$.

The $\frac{n(n-1)}{2}$ quantities $p$, introduced above, are again, as in the cases of the three or the four dimensions, the components of rotation about the $\frac{n(n-1)}{2}$ faces of the *moving polyhedron* which we have considered above. It is very easy to show this in a manner quite analogous to that of Mr. Cole* (for the space of four dimensions), and we can again express the cosines of these faces in terms of the cosines of the axes, but I omit writing the demonstration, as it is not necessary in the following lines,

8. Proceeding now in the same manner as M. Darboux and Professor Craig, we find immediately that the system of differential equations of the first order,

$$\frac{dA_i}{dt} = \sum_{\lambda=i+1}^{n} A_\lambda p_{i\lambda} - \sum_{\tau=1}^{i-1} A_\tau p_{\tau i}, \tag{7}$$
$$(i = 1, 2, 3, \ldots, n)$$

which the $n$ groups of cosines must satisfy, has always one, and *only* one, solution,

* This Journal, vol. XII, pp. 191 et seq. (1890).

when the initial values of the cosines are given; if, further, $A_1, A_2, \ldots, A_n$; $A'_1, A'_2, \ldots, A'_n$ are two systems of solutions of (7), the quantities

$$\sum_{i=1}^{n} A_i^2, \ \sum_{i=1}^{n} A_i A'_i, \ \sum_{i=1}^{n} A_i'^2$$

will be constant. We can thus repeat all the reasoning of M. Darboux and show that, when the rotations are given functions of the time ($t$), the motion is entirely determined; only the position of the fixed axes $OX_1 X_2 \ldots X_n$ is arbitrary; and, further, that, if we have $n-1$ integrals $A_1^{(0)}, A_2^{(0)}, \ldots, A_n^{(0)}$; $A_1^{(1)}, A_2^{(1)}, \ldots, A_n^{(1)}$; etc.; $A_1^{(n-2)}, A_2^{(n-2)}, \ldots, A_n^{(n-2)}$, the general integral $A_i$ will be given by the equations

$$\sum_{i=1}^{n} A_i^2 = \text{const.}, \ \sum_{i=1}^{n} A_i A_i^{(0)} = \text{const.}, \ \sum_{i=1}^{n} A_i A_i^{(1)} = \text{const.}, \ldots, \ \sum_{i=1}^{n} A_i A_i^{(n-2)} = \text{const.},$$

or by the following equations, which are all linear in $A_i$ (method of M. Cosserat mentioned by Professor Craig),

$$\sum_{i=1}^{n} A_i A_i^{(0)} = \text{const.}, \ \sum_{i=1}^{n} A_i A_i^{(1)} = \text{const.}, \ldots, \ \sum_{i=1}^{n} A_i A_i^{(n-2)} = \text{const.},$$

$$\begin{vmatrix} A_1 & A_2 & \ldots & A_n \\ A_1^{(0)} & A_2^{(0)} & \ldots & A_n^{(0)} \\ A_1^{(1)} & A_2^{(1)} & \ldots & A_n^{(1)} \\ \cdots & \cdots & \cdots & \cdots \\ A_1^{(n-2)} & A_2^{(n-2)} & \ldots & A_n^{(n-2)} \end{vmatrix} = \text{const.}$$

9. Since the $A$'s are connected by the relation

$$\sum_{i=1}^{n} A_i^2 = \text{const., or (dividing by a convenient constant)} \sum_{i=1}^{n} A_i^2 = 1,$$

we can express them in terms of only $n-1$ variables; we put for this purpose, with Professor Craig,

$$A_1 = \frac{2\Lambda_1}{k^2+1}, \quad A_2 = \frac{2\Lambda_2}{k^2+1}, \ldots, A_{n-1} = \frac{2\Lambda_{n-1}}{k^2+1}, \quad A_n = \frac{k^2-1}{k^2+1},$$

$$\left(k^2 = \sum_{i=1}^{n-1} \Lambda_i^2\right),$$

and we find, after some reductions, the following system containing one equation and one variable less than the system (7).

$$
\left.
\begin{aligned}
\frac{d\Lambda_1}{dt} &= \sum_{\lambda=2}^{n-1} \Lambda_\lambda p_{1\lambda} + \frac{k^2-1}{2} p_{1n} - \Lambda_1 \sum_{\tau=1}^{n-1} \Lambda_\tau p_{\tau n}, \\
\frac{d\Lambda_2}{dt} &= \sum_{\lambda=3}^{-1} \Lambda_\lambda p_{2\lambda} + \frac{k^2-1}{2} p_{2n} - \Lambda_2 \sum_{\tau=1}^{n-1} \Lambda_\tau p_{\tau n} - \Lambda_1 p_{12}, \\
\frac{d\Lambda_3}{dt} &= \sum_{\lambda=4}^{n-1} \Lambda_\lambda p_{3\lambda} + \frac{k^2-1}{2} p_{3n} - \Lambda_3 \sum_{\tau=1}^{n-1} \Lambda_\tau p_{\tau n} - \sum_{\tau=1}^{2} \Lambda_\tau p_{\tau 3}, \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\frac{d\Lambda_{n-3}}{dt} &= \sum_{\lambda=n-2}^{n-1} \Lambda_\lambda p_{n-3,\lambda} + \frac{k^2-1}{2} p_{n-3n} - \Lambda_{n-3} \sum_{\tau=1}^{n-1} \Lambda_\tau p_{\tau n} - \sum_{\tau=1}^{n-4} \Lambda_\tau p_{\tau n-3}, \\
\frac{d\Lambda_{n-2}}{dt} &= \Lambda_{n-1} p_{n-2,n-1} + \frac{k^2-1}{2} p_{n-2n} - \Lambda_{n-2} \sum_{\tau=1}^{n-1} \Lambda_\tau p_{\tau n} - \sum_{\tau=1}^{n-3} \Lambda_\tau p_{\tau n-2}, \\
\frac{d\Lambda_{n-1}}{dt} &= \frac{k^2-1}{2} p_{n-1n} - \Lambda_{n-1} \sum_{\tau=1}^{n-1} \Lambda_\tau p_{\tau n} - \sum_{\tau=1}^{n-2} \Lambda_\tau p_{\tau n-1}.
\end{aligned}
\right\} \quad (8)
$$

We can, of course, apply also the substitution of Mr. J. Eiesland,*

$$
A_1 = \frac{\Lambda_1}{\sqrt{k^2+1}}, \quad A_2 = \frac{\Lambda_2}{\sqrt{k^2+1}}, \; \ldots, \; A_{n-1} = \frac{\Lambda_{n-1}}{\sqrt{k^2+1}}, \quad A_n = \frac{1}{\sqrt{k^2+1}};
$$

$$
\left(k^2 = \sum_{i=1}^{n-1} \Lambda_i^2\right)
$$

we find, then, the following system of $n-1$ differential equations, equivalent, of course, to the system (8), but a little simplified,

$$
\left.
\begin{aligned}
\frac{d\Lambda_1}{dt} &= \sum_{\lambda=2}^{n-1} \Lambda_\lambda p_{1\lambda} + p_{1n} + \Lambda_1 \sum_{\tau=1}^{n-1} \Lambda_\tau p_{\tau n}, \\
\frac{d\Lambda_2}{dt} &= \sum_{\lambda=3}^{n-1} \Lambda_\lambda p_{2\lambda} + p_{2n} + \Lambda_2 \sum_{\tau=1}^{n-1} \Lambda_\tau p_{\tau n} - \Lambda_1 p_{12}, \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\frac{d\Lambda_{n-2}}{dt} &= \Lambda_{n-1} p_{n-2,n-1} + p_{n-2,n} + \Lambda_{n-2} \sum_{\tau=1}^{n-1} \Lambda_\tau p_{\tau n} - \sum_{\tau=1}^{n-3} \Lambda_\tau p_{\tau n-2}, \\
\frac{d\Lambda_{n-1}}{dt} &= p_{n-1n} + \Lambda_{n-1} \sum_{\tau=1}^{n-1} \Lambda_\tau p_{\tau n} - \sum_{\tau=1}^{n-2} \Lambda_\tau p_{\tau n-1},
\end{aligned}
\right\} \quad (8')
$$

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* This Journal, vol. XX, pp. 245 et seq. (1898).

Each of the two systems (8) and (8′) (or every other system similarly found), constitutes the generalization of the equation of Riccati in the case of $n$-dimensional spaces. It will be easy to study these systems in a manner analogous to that of Mr. Eiesland (*loc. cit.*). Perhaps I shall find occasion to recur to this subject.

10. An application of the formulæ found above is the case of the motion of the *principal* $\frac{n(n-1)}{2}$*-hedron* of a curve in the space of $n$ dimensions. $n-1$ of the $p$'s are, in this case, equal to the $n-1$ curvatures of the curve, all the others being equal to zero. (See on this subject: Ernesto Cesàro, "Geometria Intrinseca," pp. 226 *et seq.*)

11. Suppose, now, that the origin of the moving axes is not fixed; we shall then only introduce the components, relative to the moving axes $(\xi_1, \ldots, \xi_n)$, of the velocity of this origin, and we find the following $n$ equations

$$\frac{dX_1^{(0)}}{dt} = \sum_{i=1}^{n} \alpha_i \xi_i, \quad \frac{dX_2^{(0)}}{dt} = \sum_{i=1}^{n} \beta_i \xi_i, \ldots, \frac{dX_n^{(0)}}{dt} = \sum_{i=1}^{n} \nu_i \xi_i, \tag{9}$$

$X_i^{(0)}$ being the coordinates of the moving origin in the system $OX_1 X_2 \ldots X_n$. The integration of this system is again always reduced to quadratures, since the coefficients $\alpha_i, \beta_i, \ldots, \nu_i$ are supposed known as functions of the time from the system (6) and $\xi_i$ are given functions of the $t$.

12. Instead of the system (7) between the cosines of the axes and the rotations, we can find another system of $\frac{n(n-1)}{2}$ equations between the direction cosines $P$ of the faces of the moving polyhedron and the rotations. But the integration of this system can be led back to that of (7); and the relations which this second system gives between the $P$'s and the $p$'s, or between the $A$'s

and the $p$'s (for the $P$'s are given expressions of the $A$'s), must be, of course, the same as those of the system (7). So I omit writing them.

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13. I shall now consider the case of a two-parametric displacement. Denoting by $p$ the rotations depending on $u$ alone, and by $p'$ those depending on $v$ alone, we obviously have the following equations connecting the $p$'s, the $p'$'s and the cosines:

$$\left(\frac{\partial^2 \alpha_i}{\partial u\,\partial v}\equiv\right)\sum_{\lambda=i+1}^{n}\alpha_\lambda\frac{\partial p_{i\lambda}}{\partial v}+\sum_{\lambda=i+1}^{n}p_{i\lambda}\frac{\partial\alpha_\lambda}{\partial v}-\sum_{\tau=1}^{i-1}\alpha_\tau\frac{\partial p_{\tau i}}{\partial v}-\sum_{\tau=1}^{i-1}p_{\tau i}\frac{\partial\alpha_\tau}{\partial v}$$
$$=\sum_{\lambda=i+1}^{n}\alpha_\lambda\frac{\partial p'_{i\lambda}}{\partial u}+\sum_{\lambda=i+1}^{n}p'_{i\lambda}\frac{\partial\alpha_\lambda}{\partial u}-\sum_{\tau=1}^{i-1}\alpha_\tau\frac{\partial p'_{\tau i}}{\partial u}-\sum_{\tau=1}^{i-1}p'_{\tau i}\frac{\partial\alpha_\tau}{\partial u}\left(\equiv\frac{\partial^2\alpha_i}{\partial v\,\partial u}\right), \tag{10}$$

or, by the equations (6),

$$\left.\begin{aligned}&\sum_{\lambda=i+1}^{n}\alpha_\lambda\left(\frac{\partial p_{i\lambda}}{\partial v}-\frac{\partial p'_{i\lambda}}{\partial u}\right)-\sum_{\tau=1}^{i-1}\alpha_\tau\left(\frac{\partial p_{\tau i}}{\partial v}-\frac{\partial p'_{\tau i}}{\partial u}\right)\\&\qquad=\sum_{\lambda=i+1}^{n}p'_{i\lambda}\Big(\sum_{l=\lambda+1}^{n}\alpha_l\,p_{\lambda l}-\sum_{t=1}^{\lambda-1}\alpha_t p_{t\lambda}\Big)-\sum_{\tau=1}^{i-1}p'_{\tau i}\Big(\sum_{l=\tau+1}^{n}\alpha_l p_{\tau l}-\sum_{t=1}^{\tau-1}\alpha_t\,p_{t\tau}\Big)-\\&\qquad-\sum_{\lambda=i+1}^{n}p_{i\lambda}\Big(\sum_{l=\lambda+1}^{n}\alpha_l p'_{\lambda l}-\sum_{t=1}^{\lambda-1}\alpha_t\,p'_{t\lambda}\Big)+\sum_{\tau=1}^{i-1}p_{\tau i}\Big(\sum_{l=\tau+1}^{n}\alpha_l p'_{\tau l}-\sum_{t=1}^{\tau-1}\alpha_t p'_{t\tau}\Big),\end{aligned}\right\}\tag{10'}$$

where $i=1,\ 2,\ 3,\ \ldots.,\ n$.

As, now, we have similar equations for the $\beta,\ \gamma,\ \ldots.,\ \nu$, we can multiply them by $\alpha_1,\ \beta_1,\ \ldots.,\ \nu_1$ and add, or by $\alpha_2,\ \beta_2,\ \ldots.,\ \nu_2$ and add, etc., by $\alpha_n,\ \beta_n,\ \ldots.,\ \nu_n$ and add; we find thus the following $\frac{n(n-1)}{2}$ *fundamental equations of the two-parametric displacements*, connecting the $p$'s and the $p'$'s

$$
\left.
\begin{aligned}
\frac{\partial p_{1i}}{\partial v}-\frac{\partial p'_{1i}}{\partial u} &= \sum_{\lambda=i+1}^{n}(p'_{i\lambda}p_{1\lambda}-p_{i\lambda}p'_{1\lambda})-\sum_{\tau=1}^{i-1}(p'_{\tau i}p_{1\tau}-p_{\tau i}p'_{1\tau}),\\
&\qquad (i=2,\ 3,\ \ldots\ldots,\ n)\\
\frac{\partial p_{23}}{\partial v}-\frac{\partial p'_{23}}{\partial u} &= \sum_{\lambda=4}^{n}(p'_{3\lambda}p_{2\lambda}-p_{3\lambda}p'_{2\lambda})+p'_{13}p_{12}-p_{13}p'_{12},\\
\frac{\partial p_{2i}}{\partial v}-\frac{\partial p'_{2i}}{\partial u} &= \sum_{\lambda=i+1}^{n}(p'_{i\lambda}p_{2\lambda}-p_{i\lambda}p'_{2\lambda})-\sum_{\tau=3}^{i-1}(p'_{\tau i}p_{2\tau}-p_{\tau i}p'_{2\tau})+p'_{1i}p_{12}-p_{1i}p'_{12},\\
&\qquad (i=4,\ 5,\ \ldots\ldots,\ n)\\
\frac{\partial p_{34}}{\partial v}-\frac{\partial p'_{34}}{\partial u} &= \sum_{\lambda=5}^{n}(p'_{4\lambda}p_{3\lambda}-p_{4\lambda}p'_{3\lambda})+\sum_{\tau=1}^{2}(p'_{\tau 4}p_{\tau 3}-p_{\tau 4}p'_{\tau 3}),\\
\frac{\partial p_{3i}}{\partial v}-\frac{\partial p'_{3i}}{\partial u} &= \sum_{\lambda=i+1}^{n}(p'_{i\lambda}p_{3\lambda}-p_{i\lambda}p'_{3\lambda})-\sum_{\tau=4}^{i-1}(p'_{\tau i}p_{3\tau}-p_{\tau i}p'_{3\tau})\\
&\qquad +\sum_{\tau=1}^{2}(p'_{\tau i}p_{\tau 3}-p_{\tau i}p'_{\tau 3}),\\
&\qquad (i=5,\ 6,\ \ldots\ldots,\ n)\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
\frac{\partial p_{k\,k+1}}{\partial v}-\frac{\partial p'_{k\,k+1}}{\partial u} &= \sum_{\lambda=k+2}^{n}(p'_{k+1\,\lambda}p_{k\lambda}-p_{k+1\,\lambda}p'_{k\lambda})+\sum_{\tau=1}^{k-1}(p'_{\tau\,k+1}p_{\tau k}-p_{\tau\,k+1}p'_{\tau k}),\\
\frac{\partial p_{ki}}{\partial v}-\frac{\partial p'_{ki}}{\partial u} &= \sum_{\lambda=i+1}^{n}(p'_{i\lambda}p_{k\lambda}-p_{i\lambda}p'_{k\lambda})-\sum_{\tau=k+1}^{i-1}(p'_{\tau i}p_{k\tau}-p_{\tau i}p'_{k\tau})\\
&\qquad +\sum_{\tau=1}^{k-1}(p'_{\tau i}p_{\tau k}-p_{\tau i}p'_{\tau k}),\\
&\qquad (i=k+2,\ k+3,\ \ldots\ldots,\ n)\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
\frac{\partial p_{n-2,\,n-1}}{\partial v}-\frac{\partial p'_{n-2,\,n-1}}{\partial u} &= p'_{n-1,\,n}p_{n-2,\,n}-p_{n-1,\,n}p'_{n-2,\,n}\\
&\qquad +\sum_{\tau=1}^{n-3}(p'_{\tau,\,n-1}p_{\tau,\,n-2}-p_{\tau,\,n-1}p'_{\tau,\,n-2}),\\
\frac{\partial p_{n-2,\,n}}{\partial v}-\frac{\partial p'_{n-2,\,n}}{\partial u} &= -p'_{n-1,\,n}p_{n-2,\,n-1}+p_{n-1,\,n}p'_{n-2,\,n-1},\\
&\qquad +\sum_{\tau=1}^{n-3}(p'_{\tau n}p_{\tau,\,n-2}-p_{\tau n}p'_{\tau,\,n-2}),\\
\frac{\partial p_{n-1,\,n}}{\partial v}-\frac{\partial p'_{n-1,\,n}}{\partial u} &= \sum_{\tau=1}^{n-2}(p'_{\tau n}p_{\tau\,n-1}-p_{\tau n}p'_{\tau\,n-1}).
\end{aligned}
\right\}\ (11)
$$

We have found these equations, equating the coefficients of $\alpha_1, \alpha_2, \ldots, \alpha_{n-1}, \alpha_n$ on the left and the right hand. The first equation is given by $\alpha_1$, $i = 2, 3, \ldots, n$ ($i = 1$ gives zero), the second by $\alpha_2$, $i = 3$ ($i = 1$ gives again an equation of the first line, that with $\alpha = \alpha_1$, $i = 2$; $i = 2$ gives zero), the third equation is given by $\alpha_2$, $i = 4, 5, \ldots, n$; etc., the equation of the order $2k - 2$ is given by $\alpha_k$, $i = k + 1$ ($i = k$ gives zero; $i = 1, 2, \ldots, k - 1$ gives equations already found), the equation of the order $2k - 1$ is given by $\alpha_k$, $i = k + 2, \ldots n$, etc. We have also, finally, an equation by $\alpha_{n-1}$, $i = n$. Thus, together $\frac{n(n-1)}{2}$ fundamental equations.

The case $n = 3$ gives the three equations of M. Darboux, and that of $n = 4$ the six equations of Professor Craig.*

14. We can introduce the $P$'s we have mentioned above, but the equations found between the $P$'s, the $p$'s and the $p'$'s are consequences of (10′), and, consequently, we do not find new relations between the $p$'s and $p'$'s.

15. Conversely, whenever the $p$'s and $p'$'s satisfy the system (11), there exists one, and only one, motion having these quantities as rotations. The reasoning is entirely analogous to that of M. Darboux.

16. The integration of the two systems of differential equations for the cosines formed like (6) will be, of course, led back, in this case, to the integration of a simultaneous system of two sets of equations formed like (8) or (8′) and containing two equations and two variables less.

17. Let, now, the moving system have no fixed point; we find in this case

$$\frac{\partial X_1^{(0)}}{\partial u} = \sum_{i=1}^{n} \alpha_i \xi_i, \quad \frac{\partial X_2^{(0)}}{\partial u} = \sum_{i=1}^{n} \beta_i \xi_i, \ldots, \quad \frac{\partial X_n^{(0)}}{\partial u} = \sum_{i=1}^{n} \nu_i \xi_i,$$

*Remark that I have put $p_{13}$ equal to the $-p_{13}$ of M. Darboux and Professor Craig, and also $p'_{13} = -p'_{13}$.

and $$\frac{\partial X_1^{(0)}}{\partial v} = \sum_{i=1}^{n} \alpha_i \xi_i', \quad \frac{\partial X_2^{(0)}}{\partial v} = \sum_{i=1}^{n} \beta_i \xi_i', \ldots\ldots, \quad \frac{\partial X_n^{(0)}}{\partial v} = \sum_{i=1}^{n} \nu_i \xi_i',$$

denoting by $\xi_i$ the components of the velocity of the origin, relative to the moving axes, when $u$ only varies, and by $\xi_i'$ those depending on $v$ alone.

We find so again the following $n$ equations connecting the rotations $p$, $p'$ and the translations $\xi$, $\xi'$,

$$\frac{\partial \xi_k}{\partial v} - \frac{\partial \xi_k'}{\partial u} = \sum_{i=1}^{k-1} (\xi_i' p_{ik} - \xi_i p_{ik}') - \sum_{i=k+1}^{n} (\xi_i' p_{ki} - \xi_i p_{ki}'), \tag{12}$$
$$(k = 1, 2, 3, \ldots\ldots, n).$$

18. We have also the $\frac{n(n-1)}{2}$ equations (11); thus: *For the general two-parametric displacements there exist* $\frac{n(n+1)}{2}$ *fundamental equations, viz.,* $\frac{n(n-1)}{2}$ *between the rotations only, and n between rotations and translations.*

19. Conversely, whenever the $\xi$'s and the $p$'s satisfy the two sets of fundamental equations (11) and (12), there exists a two-parametric displacement, and only one, with the rotations $p$ and the translations $\xi$. The reasoning is the same as that of M. Darboux, "Surf.," I., p. 67.

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20. Let us now consider the $k$-parametric displacements. It is obvious that we shall have $\frac{k(k-1)}{2}$ sets of fundamental equations formed like (11), for we can combine the $k$ variables $u_1, u_2, \ldots\ldots, u_k$, upon which the moving system depends, in $\frac{k(k-1)}{2}$ ways two at a time. In the same way, we shall find, between rotations and translations, $\frac{k(k-1)}{2}$ sets formed like (12). Thus:

*The general k-parametric displacement has* $\frac{n(n+1)\,k(k-1)}{4}$ *fundamental equations.*

Letting $p^{(1)}$ be the rotations depending on $u_1$ alone, etc.; $p^{(k)}$ those depending

on $u_k$ alone, we shall have the same set (11) at first with $p^{(1)}, p^{(2)}$ (instead of $p, p'$), then with $p^{(1)}, p^{(3)}$, etc. And analogously for the equations (12). I omit writing all these equations, because they would occupy several pages.

21. These equations give, of course, *all* the relations existing between the $p$'s (or the $p$'s and $\xi$'s), for those found between the $p$'s and $P$'s (or the $p$'s, $P$'s and $\xi$'s) do not give *new* relations.

22. The projections, relatively to the moving axes, of the absolute velocity of a point $M$ are

$$\left.\begin{array}{l}
\frac{dx_1}{dt}+\sum_{i=1}^{k}\xi_1^{(i)}\frac{du_i}{dt}-x_2\sum_{i=1}^{k}p_{12}^{(i)}\frac{du_i}{dt}-x_3\sum_{i=1}^{k}p_{13}^{(i)}\frac{du_i}{dt}-\ldots\ldots-x_n\sum_{i=1}^{k}p_{1n}^{(i)}\frac{du_i}{dt}, \\
\frac{dx_2}{dt}+\sum_{i=1}^{k}\xi_2^{(i)}\frac{du_i}{dt}+x_1\sum_{i=1}^{k}p_{12}^{(i)}\frac{du_i}{dt}-x_3\sum_{i=1}^{k}p_{23}^{(i)}\frac{du_i}{dt}-\ldots\ldots-x_n\sum_{i=1}^{k}p_{2n}^{(i)}\frac{du_i}{dt}, \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
\frac{dx_n}{dt}+\sum_{i=1}^{k}\xi_n^{(i)}\frac{du_i}{dt}+x_1\sum_{i=1}^{k}p_{1n}^{(i)}\frac{du_i}{dt}+x_2\sum_{i=1}^{k}p_{2n}^{(i)}\frac{du_i}{dt}+\ldots\ldots+x_{n-1}\sum_{i=1}^{k}p_{n-1\,n}^{(i)}\frac{du_i}{dt},
\end{array}\right\}\quad(13)$$

where the $u_1, \ldots\ldots, u_k$ are supposed arbitrary functions of another variable $t$ (the time for example).

---

23. Let us now consider the manifoldnesses of $k$ dimensions of the space of $n$ dimensions. Each of them has $\frac{k(k-1)n(n+1)}{4}$ fundamental kinematic equations between the rotations and translations of a system of axes $Mx_1 \ldots\ldots x_n$ having as origin a point $M$ of the manifoldness. The variables $u_1, \ldots\ldots u_k$ must be, of course, all independent. We consider, further, the polyhedron $Mx_1 x_2, \ldots\ldots, x_n$ in the following *special* position: The first $k$ axes $x_1, x_2, \ldots\ldots, x_k$ lie in the *tangent* linear manifoldness of $k$ dimensions (of the given manifoldness), and the other $n-k$ axes $x_{k+1}, \ldots\ldots, x_n$ lie in the *normal* linear $(n-k)$-dimensional manifoldness. It will then be

$$\xi_\sigma^{(1)} = \xi_\sigma^{(2)} = \ldots\ldots = \xi_\sigma^{(k)} = 0,$$
$$(\sigma = k+1, k+2, \ldots\ldots, n)$$

and the element of a curve lying on the $k$-dimensional manifoldness will be given by

$$\begin{aligned} ds^2 = {} & (\xi_1^{(1)}\,du_1 + \xi_1^{(2)}\,du_2 + \dots + \xi_1^{(k)}\,du_k)^2 \\ & + (\xi_2^{(1)}\,du_1 + \xi_2^{(2)}\,du_2 + \dots + \xi_2^{(k)}\,du_k)^2 \\ & + \dots + (\xi_k^{(1)}\,du_1 + \xi_k^{(2)}\,du_2 + \dots + \xi_k^{(k)}\,du_k)^2. \end{aligned}$$

The fundamental kinematic equations of the manifoldness become then a little shorter, but I omit writing them.

We can now put

$$\left.\begin{array}{l} \sum\limits_{\sigma=1}^{\sigma=k} \xi_\sigma^{(1)^2} = E_{11},\quad \sum\limits_{\sigma=1}^{\sigma=k} \xi_\sigma^{(2)^2} = E_{22},\ \dots\dots,\quad \sum\limits_{\sigma=1}^{\sigma=k} \xi_\sigma^{(k)^2} = E_{kk}, \\ \sum\limits_{\sigma=1}^{\sigma=k} \xi_\sigma^{(1)}\xi_\sigma^{(2)} = F_{12},\quad \sum\limits_{\sigma=1}^{\sigma=k} \xi_\sigma^{(2)}\xi_\sigma^{(3)} = F_{23},\ \dots\dots,\quad \sum\limits_{\sigma=1}^{\sigma=k} \xi_\sigma^{(k-1)}\xi_\sigma^{(k)} = F_{k-1,k}, \\ \sum\limits_{\sigma=1}^{\sigma=k} \xi_\sigma^{(1)}\xi_\sigma^{(3)} = F_{13},\quad \sum\limits_{\sigma=1}^{\sigma=k} \xi_\sigma^{(2)}\xi_\sigma^{(4)} = F_{24},\ \dots,\quad \sum\limits_{\sigma=1}^{\sigma=k} \xi_\sigma^{(k-2)}\xi_\sigma^{(k)} = F_{k-2,k}, \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ \dots\dots\dots\dots\dots\dots \\ \sum\limits_{\sigma=1}^{\sigma=k} \xi_\sigma^{(1)}\xi_\sigma^{(k)} = F_{1,k}; \end{array}\right\} \qquad (14)$$

then

$$\begin{aligned} ds^2 = {} & E_{11}\,du_1^2 + \dots + E_{kk}\,du_k^2 + 2F_{12}\,du_1\,du_2 + \dots + 2F_{1k}\,du_1\,du_k + \\ & + 2F_{23}\,du_2\,du_3 + \dots + 2F_{2k}\,du_2\,du_k + \dots + 2F_{k-1,k}\,du_{k-1}\,du_k, \end{aligned} \qquad (15)$$

with the discriminant

$$\Delta^2 \equiv \begin{vmatrix} E_{11} & F_{12} & F_{13} & \dots\dots & F_{1k} \\ F_{12} & E_{22} & F_{23} & \dots\dots & F_{2k} \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ F_{1\,k-1} & F_{2\,k-1} & F_{3\,k-1} & \dots E_{k-1\,k-1} & F_{k-1\,k} \\ F_{1k} & F_{2k} & F_{3k} & \dots F_{k-1\,k} & E_{kk} \end{vmatrix}$$

$$= \begin{vmatrix} \xi_1^{(1)} & \xi_2^{(1)} & \dots & \xi_k^{(1)} \\ \xi_1^{(2)} & \xi_2^{(2)} & \dots & \xi_k^{(2)} \\ \dots & \dots & \dots & \dots \\ \xi_1^{(k)} & \xi_2^{(k)} & \dots & \xi_k^{(k)} \end{vmatrix}^2.$$

24. When the $E$'s and $F$'s are given (from the analytical equations of the manifoldness),* equations (14) serve to determine the $\xi$'s as far as it is possible. For the complete determination of the $\xi$'s, it is, of course, necessary to determine the position of the axes $x_1, \ldots, x_k$ in the tangent manifoldness, viz., to give $\frac{k(k-1)}{2}$ further relations between the $\xi$'s, which, with the $\frac{k(k+1)}{2}$ relations (14), will entirely determine the $k^2$ $\xi$'s. We now shall find the cosines and the rotations in terms of the $E$'s, $F$'s as far as it is possible.

25. We find the cosines by M. Darboux's method.

First we have

$$\left.\begin{array}{l}
\sum_{\sigma=1}^{k} \xi_\sigma^{(1)} \alpha_\sigma = \frac{\partial x_1}{\partial u_1}, \quad \sum_{\sigma=1}^{k} \xi_\sigma^{(1)} \beta_\sigma = \frac{\partial x_2}{\partial u_1}, \ldots, \sum_{\sigma=1}^{k} \xi_\sigma^{(1)} \nu_\sigma = \frac{\partial x_n}{\partial u_1}, \\
\sum_{\sigma=1}^{k} \xi_\sigma^{(2)} \alpha_\sigma = \frac{\partial x_1}{\partial u_2}, \quad \sum_{\sigma=1}^{k} \xi_\sigma^{(2)} \beta_\sigma = \frac{\partial x_2}{\partial u_2}, \ldots, \sum_{\sigma=1}^{k} \xi_\sigma^{(2)} \nu_\sigma = \frac{\partial x_n}{\partial u_2}, \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\sum_{\sigma=1}^{k} \xi_\sigma^{(k)} \alpha_\sigma = \frac{\partial x_1}{\partial u_k}, \quad \sum_{\sigma=1}^{k} \xi_\sigma^{(k)} \beta_\sigma = \frac{\partial x_2}{\partial u_k}, \ldots, \sum_{\sigma=1}^{k} \xi_\sigma^{(k)} \nu_\sigma = \frac{\partial x_n}{\partial u_k},
\end{array}\right\} \quad (16)$$

$nk$ linear equations in the $nk$ first cosines $\alpha_1, \ldots, \alpha_k$; $\beta_1, \ldots, \beta_k$; etc., $\nu_1, \ldots, \nu_k$. We find thus, $\Delta$ being the determinant of the $\xi$'s,

$$\left.\begin{array}{l}
\alpha_1 = \frac{\left| \frac{\partial x_1}{\partial u_\tau} \; \xi_2^{(\tau)} \; \xi_3^{(\tau)} \; \ldots \; \xi_k^{(\tau)} \right|}{\Delta}, \quad \alpha_2 = \frac{\left| \xi_1^{(\tau)} \; \frac{\partial x_1}{\partial u_\tau} \; \xi_3^{(\tau)} \; \ldots \; \xi_k^{(\tau)} \right|}{\Delta}, \ldots \\
\beta_1 = \frac{\left| \frac{\partial x_2}{\partial u_\tau} \; \xi_2^{(\tau)} \; \xi_3^{(\tau)} \; \ldots \; \xi_k^{(\tau)} \right|}{\Delta}, \quad \beta_2 = \frac{\left| \xi_1^{(\tau)} \; \frac{\partial x_2}{\partial u_\tau} \; \xi_3^{(\tau)} \; \ldots \; \xi_k^{(\tau)} \right|}{\Delta}, \ldots \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\nu_1 = \frac{\left| \frac{\partial x_n}{\partial u_\tau} \; \xi_2^{(\tau)} \; \xi_3^{(\tau)} \; \ldots \; \xi_k^{(\tau)} \right|}{\Delta}, \quad \nu_2 = \frac{\left| \xi_1^{(\tau)} \; \frac{\partial x_n}{\partial u_\tau} \; \xi_3^{(\tau)} \; \ldots \; \xi_k^{(\tau)} \right|}{\Delta}, \ldots
\end{array}\right\} \quad (17)$$

($\tau = 1, 2, \ldots, k$ in the determinants).

---

$$*E_{jj} = \sum_{i=1}^{n} \left( \frac{\partial x_i}{\partial u_j} \right)^2, \quad F_{jl} = \sum_{i=1}^{n} \frac{\partial x_i}{\partial u_j} \frac{\partial x_i}{\partial u_l}.$$

26. There remain now the other $n(n-k)$ cosines; these are connected with themselves and with the former found $nk$ cosines by $\frac{(n-k)(n+k+1)}{2}$ relations of the form of (2′), consequently $\frac{(n-k)(n-k-1)}{2}$ of these cosines remain indeterminate. And, indeed, in this general case, the position of the polyhedron of the manifoldness is not yet entirely determined. Assuming the $\xi$'s to be all given, we only fix the position of the $k$ axes lying in the tangent linear manifoldness, but the other $n-k$ axes lying in the normal linear manifoldness are not yet entirely determined. Only in the case of $k=n-1$, viz., of the manifoldness of the highest order, is the normal manifoldness a straight line and the cosines $\alpha_n, \beta_n, \ldots, \nu_n$ are entirely determined. In the general case of a $k$-dimensional manifoldness, the simplest manner to determine the axes in the normal manifoldness is the following: There exists a linear $(n-1)$-dimensional osculating manifoldness (or space) of the given manifoldness (very easy to show), and in this $(n-1)$-dimensional space there exists again a linear $(n-2)$-dimensional osculating space of the given manifoldness, etc., there exists, finally, a linear $k$-dimensional osculating manifoldness of the given manifoldness, the *tangent* manifoldness, lying in all the other osculating spaces of higher dimension. We now assume the axis $x_n$ to be the normal to the osculating $(n-1)$-dimensional space; in this space, the axis $x_{n-1}$ to be the normal to the osculating $(n-2)$-dimensional space, etc.; finally, the axis $x_{k+1}$ to be normal in the osculating $(k+1)$-dimensional space to the $k$-dimensional osculating space.*

27. Assuming all the cosines to be so determined, we find the rotations in the following manner:

We have

$$\sum_{\alpha,\, x} \alpha_{k+1}\, d\frac{\partial x_1}{\partial u_1} = \alpha_{k+1}\sum_{\sigma=1}^{k} \xi_\sigma^{(1)}\, d\alpha_\sigma + \beta_{k+1}\sum_{\sigma=1}^{k} \xi_\sigma^{(1)}\, d\beta_\sigma + \;\ldots\; + \nu_{k+1}\sum_{\sigma=1}^{k} \xi_\sigma^{(1)}\, d\nu_\sigma,$$

---

*An example for $k<n-1$ is that of the curves in the ordinary space of three dimensions; the normal manifoldness is here the normal plane; and the cosines are entirely determined when we assume the axes $y$ and $z$ to be the *principal normal* and the *binormal* of the curve. This second straight line is the normal to the osculating plane, the first is the normal to the tangent, in this plane. A further example is the surface in the 4-dimensional space, etc. For $k=n-1$, an example constitutes the surface in the 3-dimensional space, the hypersurface in the 4-dimensional space, etc.

or

$$\sum_{a,\,x} \alpha_{k+1}\, d\,\frac{\partial x_1}{\partial u_1} = \xi_1^{(1)} \sum \alpha_{k+1}\, d\alpha_1 + \xi_2^{(1)} \sum \alpha_{k+1}\, d\alpha_2 + \cdots + \xi_k^{(1)} \sum \alpha_{k+1}\, d\alpha_k,$$

viz.:

$$\sum_{a,\,x} \alpha_{k+1}\, d\,\frac{\partial x_1}{\partial u_1} = \xi_1^{(1)}\, \Pi_{1\,k+1} + \cdots + \xi_k^{(1)}\, \Pi_{k\,k+1},$$

where

$$\Pi_{1\,k+1} = p_{1\,k+1}^{(1)}\, du_1 + p_{1\,k+1}^{(2)}\, du_2 + \cdots + p_{1\,k+1}^{(k)}\, du_k, \text{ etc.}$$

Further,

$$\sum_{a,\,x} \alpha_{k+1}\, d\,\frac{\partial x_1}{\partial u_2} = \xi_1^{(2)}\, \Pi_{1\,k+1} + \xi_2^{(2)}\, \Pi_{2\,k+1} + \cdots + \xi_k^{(2)}\, \Pi_{k\,k+1},$$

$$\sum_{a,\,x} \alpha_{k+1}\, d\,\frac{\partial x_1}{\partial u_3} = \xi_1^{(3)}\, \Pi_{1\,k+1} + \xi_2^{(3)}\, \Pi_{2\,k+1} + \cdots + \xi_k^{(3)}\, \Pi_{k\,k+1},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\sum_{a,\,x} \alpha_{k+1}\, d\,\frac{\partial x_1}{\partial u_k} = \xi_1^{(k)}\, \Pi_{1\,k+1} + \xi_2^{(k)}\, \Pi_{2\,k+1} + \cdots + \xi_k^{(k)}\, \Pi_{k\,k+1},$$

and, consequently,

$$\left.\begin{aligned}
\Pi_{1\,k+1} &= \frac{1}{\Delta}\begin{vmatrix} \sum_{a,\,x} \alpha_{k+1}\, d\,\frac{\partial x_1}{\partial u_1} & \xi_2^{(1)} & \cdots & \xi_k^{(1)} \\ \sum_{a,\,x} \alpha_{k+1}\, d\,\frac{\partial x_1}{\partial u_2} & \xi_2^{(2)} & \cdots & \xi_k^{(2)} \\ \cdots & \cdots & \cdots & \cdots \\ \sum_{a,\,x} \alpha_{k+1}\, d\,\frac{\partial x_1}{\partial u_k} & \xi_2^{(k)} & \cdots & \xi_k^{(k)} \end{vmatrix}, \\
&(\Delta \text{ being the determinant of the } \xi\text{'s}), \text{ etc.,} \\
\Pi_{k\,k+1} &= \frac{1}{\Delta}\begin{vmatrix} \xi_1^{(1)} & \xi_2^{(1)} & \cdots & \xi_{k-1}^{(1)} & \sum_{a,\,x} \alpha_{k+1}\, d\,\frac{\partial x_1}{\partial u_1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \xi_1^{(k)} & \xi_2^{(k)} & \cdots & \xi_{k-1}^{(k)} & \sum_{a,\,x} \alpha_{k+1}\, d\,\frac{\partial x_1}{\partial u_k} \end{vmatrix}.
\end{aligned}\right\} \quad (18)$$

To find the $p^{(1)}_{1\,k+1}, p^{(2)}_{1\,k+1}, \ldots, p^{(k)}_{1\,k+1}; \ldots, p^{(1)}_{k\,k+1}, p^{(2)}_{k\,k+1}, \ldots, p^{(k)}_{k\,k+1}$, it is further only necessary to equate the coefficients of $du_1, du_2, \ldots, du_k$ on the left-hand and the right-hand side. We have thus expressed these $p$'s in terms of the $\xi$'s, the second derivatives of the $x$'s and the $\alpha_{k+1}, \ldots, \nu_{k+1}$ *only* (their forms (4) contain also the $\alpha_1, \ldots, \alpha_k$, etc., $\nu_1, \ldots, \nu_k$).

We find in a quite analogous manner the

$$\begin{array}{l}\Pi_{1\,k+2}, \ldots, \Pi_{k\,k+2}, \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ \Pi_{1n}, \quad \ldots, \Pi_{kn},\end{array}$$

and, consequently, also the $p^{(1)}_{1\,k+2}, \ldots, p^{(k)}_{1\,k+2}$; $p^{(1)}_{2\,k+2}, \ldots, p^{(k)}_{2\,k+2}$; etc. $p^{(1)}_{k\,k+2}, \ldots, p^{(k)}_{k\,k+2}$, etc., etc. $p^{(1)}_{1n}, \ldots, p^{(k)}_{1n}$, etc., $p^{(1)}_{kn}, \ldots, p^{(k)}_{kn}$. All these $p$'s we can find also in another way; it suffices to introduce in their expressions $p^{(1)}_{1\,k+1} = \sum \alpha_{k+1} \dfrac{\partial \alpha_1}{\partial u_1}$, etc., the values of $\dfrac{\partial \alpha_1}{\partial u_1}$, etc., from (17); the expressions so found, are, indeed, the same as those found from (18), etc., after some reductions.

28. Further,

$$\left.\begin{array}{l}
p^{(i)}_{k+1\,k+2} = \sum \alpha_{k+2} \dfrac{\partial \alpha_{k+1}}{\partial u_i}, \quad p^{(i)}_{k+1\,k+3} = \sum \alpha_{k+3} \dfrac{\partial \alpha_{k+1}}{\partial u_i}, \\
\qquad\qquad p^{(i)}_{k+1\,k+4} = \sum \alpha_{k+4} \dfrac{\partial \alpha_{k+1}}{\partial u_i}, \quad \ldots, \quad p^{(i)}_{k+1\,n} = \sum \alpha_n \dfrac{\partial \alpha_{k+1}}{\partial u_i}, \\
p^{(i)}_{k+2\,k+3} = \sum \alpha_{k+3} \dfrac{\partial \alpha_{k+2}}{\partial u_i}, \quad p^{(i)}_{k+2\,k+4} = \sum \alpha_{k+4} \dfrac{\partial \alpha_{k+2}}{\partial u_i}, \ldots, \\
\qquad\qquad\qquad\qquad p^{(i)}_{k+2\,n} = \sum \alpha_n \dfrac{\partial \alpha_{k+2}}{\partial u_i}, \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
\ldots\ldots\ldots\ldots\ldots \\
\qquad\qquad\qquad\qquad p^{(i)}_{n-1\,n} = \sum \alpha_n \dfrac{\partial \alpha_{n-1}}{\partial u_i}
\end{array}\right\} \quad (19)$$

$(i = 1, 2, \ldots, k)$. It is *impossible* to express these rotations by the $\xi$'s, etc.

29. To find the last rotations, we use the formulæ

$$\sum_x \frac{\partial x_1}{\partial u_2} \cdot \frac{\partial}{\partial u_1}\left(\frac{\partial x_1}{\partial u_1}\right) = \sum_{\sigma=1}^{k} \xi_\sigma^{(2)} \alpha_\sigma \left(\sum_{\sigma=1}^{k} \xi_\sigma^{(1)} \frac{\partial \alpha_\sigma}{\partial u_1} + \sum_{\sigma=1}^{k} \alpha_\sigma \frac{\partial \xi_\sigma^{(1)}}{\partial u_1}\right) + \ldots\ldots +$$

$$+ \sum_{\sigma=1}^{k} \xi_\sigma^{(2)} \nu_\sigma \left(\sum_{\sigma=1}^{k} \xi_\sigma^{(1)} \frac{\partial \nu_\sigma}{\partial u_1} + \sum_{\sigma=1}^{k} \nu_\sigma \frac{\partial \xi_\sigma^{(1)}}{\partial u_1}\right)$$

$$= - \xi_1^{(2)} \xi_2^{(1)} p_{12}^{(1)} - \xi_1^{(2)} \xi_3^{(1)} p_{13}^{(1)} - \ldots\ldots - \xi_1^{(2)} \xi_k^{(1)} p_{1k}^{(1)} +$$

$$+ \xi_2^{(2)} \xi_1^{(1)} p_{12}^{(1)} - \xi_2^{(2)} \xi_3^{(1)} p_{23}^{(1)} - \ldots\ldots - \xi_2^{(2)} \xi_k^{(1)} p_{2k}^{(1)} +$$

$$+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots +$$

$$+ \xi_k^{(2)} \xi_1^{(1)} p_{1k}^{(1)} + \xi_k^{(2)} \xi_2^{(1)} p_{2k}^{(1)} + \ldots\ldots + \xi_k^{(2)} \xi_{k-1}^{(1)} p_{k-1k}^{(1)} + \ldots\ldots$$

$$= p_{12}^{(1)} (\xi_2^{(2)} \xi_1^{(1)} - \xi_1^{(2)} \xi_2^{(1)}) + p_{13}^{(1)} (\xi_3^{(2)} \xi_1^{(1)} - \xi_1^{(2)} \xi_3^{(1)}) + \ldots\ldots +$$

$$+ p_{k-1k}^{(1)} (\xi_k^{(2)} \xi_{k-1}^{(1)} - \xi_{k-1}^{(2)} \xi_k^{(1)}) + \xi_1^{(2)} \frac{\partial \xi_1^{(1)}}{\partial u_1} + \ldots\ldots + \xi_k^{(2)} \frac{\partial \xi_k^{(1)}}{\partial u_1}.$$

Forming, further, the similar expressions for

$$\sum_x \frac{\partial x_1}{\partial u_3} \frac{\partial}{\partial u_1}\left(\frac{\partial x_1}{\partial u_1}\right), \ldots\ldots, \sum_x \frac{\partial x_1}{\partial u_k} \frac{\partial}{\partial u_1}\left(\frac{\partial x_1}{\partial u_1}\right),$$

$$\sum_x \frac{\partial x_1}{\partial u_3} \frac{\partial}{\partial u_1}\left(\frac{\partial x_1}{\partial u_2}\right), \ldots\ldots, \sum_x \frac{\partial x_1}{\partial u_k} \frac{\partial}{\partial u_1}\left(\frac{\partial x_1}{\partial u_2}\right),$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\sum_x \frac{\partial x_1}{\partial u_k} \frac{\partial}{\partial u_1}\left(\frac{\partial x_1}{\partial u_{k-1}}\right),$$

we have $\frac{k(k-1)}{2}$ linear equations for the $\frac{k(k-1)}{2}$ rotations $p_{12}^{(1)}, \ldots\ldots, p_{1k}^{(1)}, \ldots\ldots,$ $p_{k-1k}^{(1)}$. And, putting in the expressions written above $\frac{\partial}{\partial u_2}, \ldots\ldots, \frac{\partial}{\partial u_k}$, instead of $\frac{\partial}{\partial u_1}$, we get the other rotations $p_{12}^{(2)}, \ldots\ldots, p_{1k}^{(2)}, \ldots\ldots, p_{k-1k}^{(2)}$, etc., etc., $p_{12}^{(k)}, \ldots\ldots,$ $p_{1k}^{(k)}, \ldots\ldots, p_{k-1k}^{(k)}$. Thus for example:

$$
p^{(1)}_{12} = \frac{1}{\Omega}\left|\begin{array}{lllll}
\sum\limits_x \dfrac{\partial x_1}{\partial u_2}\,\dfrac{\partial^2 x_1}{\partial u_1^2} & \xi^{(2)}_3\,\xi^{(1)}_1 - \xi^{(2)}_1\,\xi^{(1)}_3 & \cdots\cdots & \xi^{(2)}_k\,\xi^{(1)}_{k-1} - \xi^{(2)}_{k-1}\,\xi^{(1)}_k \\
\sum\limits_x \dfrac{\partial x_1}{\partial u_3}\,\dfrac{\partial^2 x_1}{\partial u_1^2} & \xi^{(3)}_3\,\xi^{((1)}_1 - \xi^{(3)}_1\,\xi^{(1)}_3 & \cdots\cdots & \xi^{(3}_k\,\xi^{(1)}_{k-1} - \xi^{(3)}_{k-1}\,\xi^{(1)}_k \\
\cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \\
\sum\limits_x \dfrac{\partial x_1}{\partial u_k}\,\dfrac{\partial^2 x_1}{\partial u_1^2} & \xi^{(k)}_3\,\xi^{(1)}_1 - \xi^{(k)}_1\,\xi^{(1)}_3 & \cdots\cdots & \xi^{(k)}_k\,\xi^{(1)}_{k-1} - \xi^{(k)}_{k-1}\,\xi^{(1)}_k \\
\sum\limits_x \dfrac{\partial x_1}{\partial u_3}\,\dfrac{\partial^2 x}{\partial u_2\,\partial u_1} & \xi^{(3)}_3\,\xi^{(2)}_1 - \xi^{(3)}_1\,\xi^{(2)}_3 & \cdots\cdots & \xi^{(3)}_k\,\xi^{(2)}_{k-1} - \xi^{(3)}_{k-1}\,\xi^{(2)}_k \\
\sum\limits_x \dfrac{\partial x_1}{\partial u_4}\,\dfrac{\partial^2 x}{\partial u_2\,\partial u_1} & \xi^{(4)}_3\,\xi^{(2)}_1 - \xi^{(4)}_1\,\xi^{(2)}_3 & \cdots\cdots & \xi^{(4)}_k\,\xi^{(2)}_{k-1} - \xi^{(4)}_{k-1}\,\xi^{(2)}_k \\
\cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \\
\sum\limits_x \dfrac{\partial x_1}{\partial u_k}\,\dfrac{\partial^2 x}{\partial u_2\,\partial u_1} & \xi^{(k)}_3\,\xi^{(2)}_1 - \xi^{(k)}_1\,\xi^{(2)}_3 & \cdots\cdots & \xi^{(k)}_k\,\xi^{(2)}_{k-1} - \xi^{(k)}_{k-1}\,\xi^{(2)}_k \\
\cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \\
\cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \\
\sum\limits_x \dfrac{\partial x_1}{\partial u_k}\,\dfrac{\partial^2 x}{\partial u_{k-1}\,\partial u_1} & \xi^{(k)}_3\,\xi^{(k-1)}_1 - \xi^{(k)}_1\,\xi^{(k-1)}_3 & \cdots\cdots & \xi^{(k)}_k\,\xi^{(k-1)}_{k-1} - \xi^{(k)}_{k-1}\,\xi^{(k-1)}_k
\end{array}\right|
$$

+ an expression of only the $\xi$'s. ($\Omega$ being the determinant of the system.)

And after some reductions we get

$$
p^{(1)}_{12} = \frac{1}{2\Omega}\left|\begin{array}{lll}
-\frac{1}{2}\dfrac{\partial E_{11}}{\partial u_2} + \sum\limits_{\sigma=1}^{k}\xi^{(1)}_\sigma\,\dfrac{\partial \xi^{(2)}_\sigma}{\partial u_1} & \xi^{(2)}_3\,\xi^{(1)}_1 - \xi^{(2)}_1\,\xi^{(1)}_3 & \cdots \\
-\frac{1}{2}\dfrac{\partial E_{11}}{\partial u_3} + \sum\limits_{\sigma=1}^{k}\xi^{(1)}_\sigma\,\dfrac{\partial \xi^{(3)}_\sigma}{\partial u_1} & \cdots\cdots & \cdots \\
\cdots\cdots & \cdots\cdots & \cdots \\
-\frac{1}{2}\dfrac{\partial E_{11}}{\partial u_k} + \sum\limits_{\sigma=1}^{k}\xi^{(1)}_\sigma\,\dfrac{\partial \xi^{(k)}_\sigma}{\partial u_1} & \cdots\cdots & \cdots \\
\frac{1}{2}\left(\dfrac{\partial F_{23}}{\partial u_1} - \dfrac{\partial F_{12}}{\partial u_3} + \dfrac{\partial F_{13}}{\partial u_2}\right) - \sum\limits_{\sigma=1}^{k}\xi^{(3)}_\sigma\,\dfrac{\partial \xi^{(2)}_\sigma}{\partial u_1} & \cdots\cdots & \cdots \\
\frac{1}{2}\left(\dfrac{\partial F_{24}}{\partial u_1} - \dfrac{\partial F_{12}}{\partial u_4} + \dfrac{\partial F_{14}}{\partial u_2}\right) - \sum\limits_{\sigma=1}^{k}\xi^{(4)}_\sigma\,\dfrac{\partial \xi^{(2)}_\sigma}{\partial u_1} & \cdots\cdots & \cdots \\
\cdots\cdots & \cdots\cdots & \cdots \\
\frac{1}{2}\left(\dfrac{\partial F_{2k}}{\partial u_1} - \dfrac{\partial F_{12}}{\partial u_k} + \dfrac{\partial F_{1k}}{\partial u_2}\right) - \sum\limits_{\sigma=1}^{k}\xi^{(k)}_\sigma\,\dfrac{\partial \xi^{(2)}_\sigma}{\partial u_1} & \cdots\cdots & \cdots \\
\cdots\cdots & \cdots\cdots & \cdots \\
\frac{1}{2}\left(\dfrac{\partial F_{k-1\,k}}{\partial u_1} - \dfrac{\partial F_{1\,k-1}}{\partial u_k} + \dfrac{\partial F_{1k}}{\partial u_{k-1}}\right) - \sum\limits_{\sigma=1}^{k}\xi^{(k)}_\sigma\,\dfrac{\partial \xi^{(k-1)}_\sigma}{\partial u_1} & \cdots\cdots & \cdots
\end{array}\right| ; \quad (20)
$$

the $p_{12}^{(1)}$ (and, consequently, also the $p_{13}^{(1)}, \ldots, p_{k-1k}^{(1)}$) can be expressed by only the $\xi$'s and the $E$'s and $F$'s, analogously to the rotations, in the case of the surfaces of the 3-dimensional space. And all the other rotations $p_{12}^{(i)}, \ldots,$ $p_{1k}^{(i)}, \ldots, p_{k-1k}^{(i)}$ can also be expressed by the same quantities. Thus, for example,

$$p_{12}^{(2)} = \frac{1}{\Omega} \left| \begin{array}{lll} \frac{1}{2}\frac{\partial E_{22}}{\partial u_1} - \sum_{\sigma=1}^{k} \xi_\sigma^{(2)} \frac{\partial \xi_\sigma^{(1)}}{\partial u_2} & \xi_3^{(2)}\xi_1^{(1)} - \xi_1^{(2)}\xi_3^{(1)} & \cdots \\ \frac{1}{2}\left(\frac{\partial F_{23}}{\partial u_1} - \frac{\partial F_{12}}{\partial u_3} + \frac{\partial F_{13}}{\partial u_2}\right) - \sum_{\sigma=1}^{k} \xi_\sigma^{(3)} \frac{\partial \xi_\sigma^{(1)}}{\partial u_2} & \xi_3^{(3)}\xi_1^{(1)} - \xi_1^{(3)}\xi_3^{(1)} & \cdots \\ \frac{1}{2}\left(\frac{\partial F_{24}}{\partial u_1} - \frac{\partial F_{12}}{\partial u_4} + \frac{\partial F_{14}}{\partial u_2}\right) - \sum_{\sigma=1}^{k} \xi_\sigma^{(4)} \frac{\partial \xi_\sigma^{(1)}}{\partial u_2} & \xi_3^{(4)}\xi_1^{(1)} - \xi_1^{(4)}\xi_3^{(1)} & \cdots \\ \cdots\cdots\cdots & \cdots\cdots & \cdots \\ \frac{1}{2}\left(\frac{\partial F_{2k}}{\partial u_1} - \frac{\partial F_{12}}{\partial u_k} + \frac{\partial F_{1k}}{\partial u_2}\right) - \sum_{\sigma=1}^{k} \xi_\sigma^{(k)} \frac{\partial \xi_\sigma^{(1)}}{\partial u_2} & \xi_3^{(k)}\xi_1^{(1)} - \xi_1^{(k)}\xi_3^{(1)} & \cdots \\ -\frac{1}{2}\frac{\partial E_{22}}{\partial u_3} + \sum_{\sigma=1}^{k} \xi_\sigma^{(2)} \frac{\partial \xi_\sigma^{(3)}}{\partial u_2} & \xi_3^{(3)}\xi_1^{(2)} - \xi_1^{(3)}\xi_3^{(2)} & \cdots \\ -\frac{1}{2}\frac{\partial E_{22}}{\partial u_4} + \sum_{\sigma=1}^{k} \xi_\sigma^{(2)} \frac{\partial \xi_\sigma^{(4)}}{\partial u_2} & \xi_3^{(4)}\xi_1^{(2)} - \xi_1^{(4)}\xi_3^{(2)} & \cdots \\ \cdots\cdots\cdots & \cdots\cdots & \cdots \\ -\frac{1}{2}\frac{\partial E_{22}}{\partial u_k} + \sum_{\sigma=1}^{k} \xi_\sigma^{(2)} \frac{\partial \xi_\sigma^{(k)}}{\partial u_2} & \xi_3^{(k)}\xi_1^{(2)} - \xi_1^{(k)}\xi_3^{(2)} & \cdots \\ \frac{1}{2}\left(\frac{\partial F_{34}}{\partial u_2} - \frac{\partial F_{23}}{\partial u_4} + \frac{\partial F_{24}}{\partial u_3}\right) - \sum_{\sigma=1}^{k} \xi_\sigma^{(4)} \frac{\partial \xi_\sigma^{(3)}}{\partial u_2} & \xi_3^{(4)}\xi_1^{(3)} - \xi_1^{(4)}\xi_3^{(3)} & \cdots \\ \cdots\cdots\cdots & \cdots\cdots & \cdots \\ \cdots\cdots\cdots & \cdots\cdots & \cdots \\ \frac{1}{2}\left(\frac{\partial F_{k-1k}}{\partial u_2} - \frac{\partial F_{2\,k-1}}{\partial u_k} + \frac{\partial F_{2k}}{\partial u_{k-1}}\right) - \sum_{\sigma=1}^{k} \xi_\sigma^{(k)} \frac{\partial \xi_\sigma^{(k-1)}}{\partial u_2} & \xi_3^{(k)}\xi_1^{(k-1)} - \xi_1^{(k)}\xi_3^{(k-1)} & \cdots \end{array} \right|, \quad (21)$$

etc., etc.

30. If the *parametric* lines:

$u_1 = \text{const.}, u_2 = \text{const.}, \ldots, u_{k-1} = \text{const.}; u_1 = \text{const.}, u_2 = \text{const.}, \ldots, u_{k-2} = \text{const.}, u_k = \text{const.}; \ldots\ldots; u_2 = \text{const.}, \ldots, u_k = \text{const.};$

are the *lines of curvature* of the given manifoldness and the axes $x_k, \ldots, x_1$ coincide with their tangents, the formulæ become a little shorter; but I shall omit this point and examine the case of $k = n - 1$, viz., the case of the mani-

foldnesses, which, in our general space, correspond exactly to the surfaces of the space of three dimensions.

31. Now

$$\left.\begin{array}{l}\sum_{\sigma=1}^{n-1}\xi_\sigma^{(1)}\alpha_\sigma=\dfrac{\partial x_1}{\partial u_1},\quad \sum_{\sigma=1}^{n-1}\xi_\sigma^{(1)}\beta_\sigma=\dfrac{\partial x_2}{\partial u_1},\ \ldots\ldots,\ \sum_{\sigma=1}^{n-1}\xi_\sigma^{(1)}\nu_\sigma=\dfrac{\partial x_n}{\partial u_1},\\ \ldots,\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\ \sum_{\sigma=1}^{n-1}\xi_\sigma^{(n-1)}\alpha_\sigma=\dfrac{\partial x_1}{\partial u_{n-1}},\quad \sum_{\sigma=1}^{n-1}\xi_\sigma^{(n-1)}\beta_\sigma=\dfrac{\partial x_2}{\partial u_{n-1}},\ \ldots\ldots,\ \sum_{\sigma=1}^{n-1}\xi_\sigma^{(n-1)}\nu_\sigma=\dfrac{\partial x_n}{\partial u_{n-1}},\end{array}\right\}\quad(22)$$

from these equations we have at once

$$\left.\begin{array}{l}\alpha_1=\dfrac{\begin{vmatrix}\dfrac{\partial x_1}{\partial u_\tau}\ \xi_2^{(\tau)}\ \xi_3^{(\tau)}\ \ldots\ldots\ \xi_{n-1}^{(\tau)}\end{vmatrix}}{D},\ \ldots\ldots,\ \alpha_{n-1}=\dfrac{\begin{vmatrix}\xi_1^{(\tau)}\ \xi_2^{(\tau)}\ \ldots\ldots\ \xi_{n-2}^{(\tau)}\ \dfrac{\partial x_1}{\partial u_\tau}\end{vmatrix}}{D},\\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\ \nu_1=\dfrac{\begin{vmatrix}\dfrac{\partial x_n}{\partial u_\tau}\ \xi_2^{(\tau)}\ \xi_3^{(\tau)}\ \ldots\ldots\ \xi_{n-1}^{(\tau)}\end{vmatrix}}{D},\ \ldots\ldots,\ \nu_{n-1}=\dfrac{\begin{vmatrix}\xi_1^{(\tau)}\ \xi_2^{(\tau)}\ \ldots\ldots\ \xi_{n-2}^{(\tau)}\ \dfrac{\partial x_n}{\partial u_\tau}\end{vmatrix}}{D},\end{array}\right\}\quad(23)$$

$D$ being the determinant of the system.

32. The other $n$ cosines $\alpha_n, \beta_n, \ldots\ldots, \nu_n$ are given by a well-known property of the orthogonal determinants *under the text*;* namely:

$$\alpha_n=\pm\begin{vmatrix}\beta_1 & \ldots\ldots & \beta_{n-1}\\ \ldots & \ldots & \ldots\\ \ldots & \ldots & \ldots\\ \nu_1 & \ldots\ldots & \nu_{n-1}\end{vmatrix}(-1)^{n-1},\ \ldots\ldots,\ \nu_n=\pm\begin{vmatrix}\alpha_1 & \ldots\ldots & \alpha_{n-1}\\ \ldots & \ldots & \ldots\\ \ldots & \ldots & \ldots\\ \mu_1 & \ldots\ldots & \mu_{n-1}\end{vmatrix},$$

each element equals its *algebraical complement* multiplied by $\pm 1$ (the determinant). It is thus (if we suppose the determinant, $= +1$, which is always possible):

$$\alpha_n=\frac{(-1)^{n-1}}{D^{n-1}}\begin{vmatrix}\begin{vmatrix}\dfrac{\partial x_2}{\partial u_\tau}\xi_2^{(\tau)} \ldots\ldots \xi_{n-1}^{(\tau)}\end{vmatrix} & \begin{vmatrix}\xi_1^{(\tau)}\dfrac{\partial x_2}{\partial u_\tau}\xi_3^{(\tau)} \ldots\ldots \xi_{n-1}^{(\tau)}\end{vmatrix} & \ldots\ldots & \begin{vmatrix}\xi_1^{(\tau)} \ldots\ldots \xi_{n-2}^{(\tau)}\dfrac{\partial x_2}{\partial u_\tau}\end{vmatrix}\\ \begin{vmatrix}\dfrac{\partial x_3}{\partial u_\tau}\xi_2^{(\tau)} \ldots\ldots \xi_{n-1}^{(\tau)}\end{vmatrix} & \begin{vmatrix}\xi_1^{(\tau)}\dfrac{\partial x_3}{\partial u_\tau}\xi_3^{(\tau)} \ldots\ldots \xi_{n-1}^{(\tau)}\end{vmatrix} & \ldots\ldots & \begin{vmatrix}\xi_1^{(\tau)} \ldots\ldots \xi_{n-2}^{(\tau)}\dfrac{\partial x_3}{\partial u_\tau}\end{vmatrix}\\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots\\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots\\ \begin{vmatrix}\dfrac{\partial x_{n-1}}{\partial u_\tau}\xi_2^{(\tau)} \ldots\ldots \xi_{n-1}^{(\tau)}\end{vmatrix} & \begin{vmatrix}\xi_1^{(\tau)}\dfrac{\partial x_{n-1}}{\partial u_\tau}\xi_3^{(\tau)} \ldots\ldots \xi_{n-1}^{(\tau)}\end{vmatrix} & \ldots\ldots & \begin{vmatrix}\xi_1^{(\tau)} \ldots\ldots \xi_{n-2}^{(\tau)}\dfrac{\partial x_{n-1}}{\partial u_\tau}\end{vmatrix}\\ \begin{vmatrix}\dfrac{\partial x_n}{\partial u_\tau}\xi_2^{(\tau)} \ldots\ldots \xi_{n-1}^{(\tau)}\end{vmatrix} & \begin{vmatrix}\xi_1^{(\tau)}\dfrac{\partial x_n}{\partial u_\tau}\xi_3^{(\tau)} \ldots\ldots \xi_{n-1}^{(\tau)}\end{vmatrix} & \ldots\ldots & \begin{vmatrix}\xi_1^{(\tau)} \ldots\ldots \xi_{n-2}^{(\tau)}\dfrac{\partial x_n}{\partial u_\tau}\end{vmatrix}\end{vmatrix};$$

* See, for example, Pascal's I Determinanti, p. 205.

but this *compound* determinant is only equal to the product

$$\begin{vmatrix} \frac{\partial x_2}{\partial u_1} & \cdots & \frac{\partial x_n}{\partial u_1} \\ \cdots & \cdots & \cdots \\ \frac{\partial x_2}{\partial u_{n-1}} & \cdots & \frac{\partial x_n}{\partial u_{n-1}} \end{vmatrix} . D^{n-2}.*$$

Consequently,

$$\alpha_n = \frac{1}{D} \frac{d(x_2, x_3, \ldots, x_n)}{d(u_1, u_2, \ldots, u_{n-1})} (-1)^{n-1}, \quad \beta_n = \frac{1}{D} \frac{d(x_1, x_3, \ldots, x_{nn})}{d(u_1, u_2, \ldots, u_{n-1})} (-1)^{n-2},$$
$$\ldots, \nu_n = \frac{1}{D} \frac{d(x_1, x_2, \ldots, x_{n-1})}{d(u_1, u_2, \ldots, u_{n-1})}. \qquad (24)$$

33. It remains to find the rotations in terms of the $\xi$'s, $E$'s and $F$'s. We get for this purpose from (22),

$$\sum_{\alpha, x} \alpha_n d \frac{\partial x_1}{\partial u_1} = \alpha_n \left[ \sum_{\sigma=1}^{n-1} \xi_\sigma^{(1)} d\alpha_\sigma + \sum_{\sigma=1}^{n-1} \alpha_\sigma d\xi_\sigma^{(1)} \right] + \beta_n \left[ \sum_{\sigma=1}^{n-1} \xi_\sigma^{(1)} d\beta_\sigma + \sum_{\sigma=1}^{n-1} \beta_\sigma d\xi_\sigma^{(1)} \right] +$$
$$+ \ldots + \nu_n \left[ \sum_{\sigma=1}^{n-1} \xi_\sigma^{(1)} d\nu_\sigma + \sum_{\sigma=1}^{n-1} \nu_\sigma d\xi_\sigma^{(1)} \right],$$

---

* This theorem, not yet observed, I think, can be easily shown. Taking the determinants A and B and $\Gamma$,

$$\mathrm{A} \equiv \begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ a_{n1} & \cdots & a_{nn} \end{vmatrix}, \qquad \mathrm{B} \equiv \begin{vmatrix} \beta_{11} & \cdots & \beta_{1n} \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ \beta_{n1} & \cdots & \beta_{nn} \end{vmatrix},$$

$$\Gamma \equiv \left\| \begin{matrix} \begin{vmatrix} \beta_{11} & a_{12} & \cdots & a_{1n} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \beta_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} & \cdots & \begin{vmatrix} a_{11} & \cdots & a_{1\,n-1} & \beta_{11} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & \cdots & a_{n\,n-1} & \beta_{n1} \end{vmatrix} \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ \begin{vmatrix} \beta_{1n} & a_{12} & \cdots & a_{1n} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \beta_{nn} & a_{n2} & \cdots & a_{nn} \end{vmatrix} & \cdots & \begin{vmatrix} a_{11} & \cdots & a_{1\,n-1} & \beta_{1n} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & \cdots & a_{n\,n-1} & \beta_{nn} \end{vmatrix} \end{matrix} \right\|,$$

the determinant $\Gamma$ can be written

$$\Gamma = \begin{vmatrix} \beta_{11} & \cdots & \beta_{1n} \\ \cdots & \cdots & \cdots \\ \beta_{n1} & \cdots & \beta_{nn} \end{vmatrix} \begin{vmatrix} \mathrm{A}_{11} & \cdots & \mathrm{A}_{1n} \\ \cdots & \cdots & \cdots \\ \mathrm{A}_{n1} & \cdots & \mathrm{A}_{nn} \end{vmatrix},$$

$\mathrm{A}_{ik}$ being the *algebraical complements* of the $a_{ik}$'s (in the determinant A); and by a well-known theorem we have consequently (Pascal, p. 43)

$$\Gamma = \mathrm{B} . \mathrm{A}^{n-1}. \qquad \text{Q. E. D.}$$

or

$$\frac{(-1)^{n-1}}{D}\begin{vmatrix} \sum_{i=1}^{n-1}\frac{\partial^2 x_1}{\partial u_1\,\partial u_i}du_i & \sum_{i=1}^{n-1}\frac{\partial^2 x_2}{\partial u_1\,\partial u_i}du_i & \cdots & \sum_{i=1}^{n-1}\frac{\partial^2 x_n}{\partial u_1\,\partial u_i}du_i \\ \frac{\partial x_1}{\partial u_1} & \frac{\partial x_2}{\partial u_1} & \cdots & \frac{\partial x_n}{\partial u_1} \\ \frac{\partial x_1}{\partial u_2} & \frac{\partial x_2}{\partial u_2} & \cdots & \frac{\partial x_n}{\partial x_2} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial x_1}{\partial u_{n-1}} & \frac{\partial x_2}{\partial u_{n-1}} & \cdots & \frac{\partial x_n}{\partial u_{n-1}} \end{vmatrix}$$

$$= \xi_1^{(1)}\Pi_{1n} + \xi_2^{(1)}\Pi_{2n} + \cdots + \xi_{n-1}^{(1)}\Pi_{n-1n},$$

where $\Pi_{1n} = p_{1n}^{(1)}\,du_1 + \cdots + p_{1n}^{(n-1)}\,du_{n-1}$, etc.

Also

$$\frac{(-1)^{n-1}}{D}\begin{vmatrix} \sum_{i=1}^{n-1}\frac{\partial^2 x_1}{\partial u_2\,\partial u_i}du_i & \cdots & \sum_{i=1}^{n-1}\frac{\partial^2 x_n}{\partial u_2\,\partial u_i}du_i \\ \frac{\partial x_1}{\partial u_1} & \cdots & \frac{\partial x_n}{\partial u_1} \\ \cdots & \cdots & \cdots \\ \frac{\partial x_1}{\partial u_{n-1}} & \cdots & \frac{\partial x_n}{\partial u_{n-1}} \end{vmatrix}$$

$$= \xi_1^{(2)}\Pi_{1n} + \xi_2^{(2)}\Pi_{2n} + \cdots + \xi_{n-1}^{(2)}\Pi_{n-1n},$$

and other $n-3$ equations with $\partial u_3$, $\xi^{(3)}$; $\partial u_4$, $\xi^{(4)}$; $\ldots$; $\partial u_{n-1}$, $\xi^{(n-1)}$. Solving them, we get

$$\Pi_{1n} = \frac{(-1)^{n-1}}{D^2}\begin{vmatrix} \overline{D}_1 & \xi_2^{(1)} & \cdots & \xi_{n-1}^{(1)} \\ \overline{D}_2 & \xi_2^{(2)} & \cdots & \xi_{n-1}^{(2)} \\ \cdots & \cdots & \cdots & \cdots \\ \overline{D}_{n-1} & \xi_2^{(n-1)} & \cdots & \xi_{n-1}^{(n-1)} \end{vmatrix}, \ \ldots,$$

$$\Pi_{n-1n} = \frac{(-1)^{n-1}}{D^2}\begin{vmatrix} \xi_1^{(1)} & \cdots & \xi_{n-2}^{(1)} & \overline{D}_1 \\ \xi_1^{(2)} & \cdots & \xi_{n-2}^{(2)} & \overline{D}_2 \\ \cdots & \cdots & \cdots & \cdots \\ \xi_1^{(n-1)} & \cdots & \xi_{n-2}^{(n-1)} & \overline{D}_{n-1} \end{vmatrix},$$

and, consequently,

$$\left.\begin{aligned}
p_{1n}^{(1)} &= \frac{(-1)^{n-1}}{D^2}\begin{vmatrix} D_{11} & \xi_2^{(1)} & \dots & \xi_{n-1}^{(1)} \\ D_{21} & \xi_2^{(2)} & \dots & \xi_{n-1}^{(2)} \\ \dots & \dots & \dots & \dots \\ D_{n-11} & \xi_2^{(n-1)} & \dots & \xi_{n-1}^{(n-1)} \end{vmatrix}, \dots\dots, \\
p_{1n}^{(n-1)} &= \frac{(-1)^{n-1}}{D^2}\begin{vmatrix} D_{1\,n-1} & \xi_2^{(1)} & \dots & \xi_{n-1}^{(1)} \\ D_{2\,n-1} & \xi_2^{(2)} & \dots & \xi_{n-1}^{(2)} \\ \dots & \dots & \dots & \dots \\ D_{n-1\,n-1} & \xi_2^{(n-1)} & \dots & \xi_{n-1}^{(n-1)} \end{vmatrix}, \\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\
p_{n-1n}^{(1)} &= \frac{(-1)^{n-1}}{D^2}\begin{vmatrix} \xi_1^{(1)} & \dots & \xi_{n-2}^{(1)} & D_{11} \\ \dots & \dots & \dots & \dots \\ \xi_1^{(n-1)} & \dots & \xi_{n-2}^{(n-1)} & D_{n-11} \end{vmatrix}, \dots\dots, \\
p_{n-1n}^{(n-1)} &= \frac{(-1)^{n-1}}{D^2}\begin{vmatrix} \xi_1^{(1)} & \dots & \xi_{n-2}^{(1)} & D_{1\,n-1} \\ \dots & \dots & \dots & \dots \\ \xi_1^{(n-1)} & \dots & \xi_{n-2}^{(n-1)} & D_{n-1\,n-1} \end{vmatrix},
\end{aligned}\right\} \tag{25}$$

where

$$D_{ik} = \begin{vmatrix} \frac{\partial^2 x_1}{\partial u_i \partial u_k} & \dots & \frac{\partial^2 x_n}{\partial u_i \partial u_k} \\ \frac{\partial x_1}{\partial u_1} & \dots & \frac{\partial x_n}{\partial u_1} \\ \dots & \dots & \dots \\ \frac{\partial x_1}{\partial u_{n-1}} & \dots & \frac{\partial x_n}{\partial u_{n-1}} \end{vmatrix}.$$

34. Further,

$$\sum_x \frac{\partial x_1}{\partial u_2} d\frac{\partial x_1}{\partial u_1} = \sum_{\sigma=1}^{n-1} \xi_\sigma^{(2)} \alpha_\sigma \left(\sum_{\sigma=1}^{n-1} \xi_\sigma^{(1)} d\alpha_\sigma + \sum_{\sigma=1}^{n-1} \alpha_\sigma\, d\xi_\sigma^{(1)}\right) + \dots + \\ + \sum_{\sigma=1}^{n-1} \xi_\sigma^{(2)} \nu_\sigma \left(\sum_{\sigma=1}^{n-1} \xi_\sigma^{(1)} d\nu_\sigma + \sum_{\sigma=1}^{n-1} \nu_\sigma\, d\xi_\sigma^{(1)}\right),$$

or

$$\left(\frac{\partial F_{12}}{\partial u_1} - \tfrac{1}{2}\frac{E_{11}}{\partial u_2}\right) du_1 + \tfrac{1}{2}\frac{\partial E_{22}}{\partial u_1} du_2 + \tfrac{1}{2}\left(\frac{\partial F_{23}}{\partial u_1} - \frac{\partial F_{13}}{\partial u_2} + \frac{\partial F_{12}}{\partial u_3}\right) du_3 +$$

$$+ \tfrac{1}{2}\left(\frac{\partial F_{24}}{\partial u_1} - \frac{\partial F_{14}}{\partial u_2} + \frac{\partial F_{12}}{\partial u_4}\right) du_4 + \ldots\ldots + \tfrac{1}{2}\left(\frac{\partial F_{2\,n-1}}{\partial u_1} - \frac{\partial F_{1\,n-1}}{\partial u_2} + \frac{\partial F_{12}}{\partial u_{n-1}}\right) du_{n-1}$$

$$= \xi_1^{(2)} d\xi_1^{(1)} + \xi_2^{(2)} d\xi_2^{(1)} + \ldots\ldots + \xi_{n-1}^{(2)} d\xi_{n-1}^{(1)} +$$

$$+ \xi_1^{(2)} (- \xi_2^{(1)} \Pi_{12} - \xi_3^{(1)} \Pi_{13} - \ldots\ldots - \xi_{n-1}^{(1)} \Pi_{1\,n-1}) +$$

$$+ \xi_2^{(2)} (+ \xi_1^{(1)} \Pi_{12} - \xi_3^{(1)} \Pi_{23} - \ldots\ldots - \xi_{n-1}^{(1)} \Pi_{2\,n-1}) +$$

$$+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots +$$

$$+ \xi_{n-1}^{(2)} (\xi_1^{(1)} \Pi_{1\,n-1} + \xi_2^{(1)} \Pi_{2\,n-1} + \ldots\ldots + \xi_{n-2}^{(1)} \Pi_{n-2\,n-1})$$

$$= \sum_{\sigma=1}^{n-1} \xi_\sigma^{(2)} d\xi_\sigma^{(1)} + \Pi_{12} (\xi_2^{(2)} \xi_1^{(1)} - \xi_1^{(2)} \xi_2^{(1)}) +$$

$$+ \Pi_{13} (\xi_{(3)}^{(2)} \xi_{(1)}^{(1)} - \xi_1^{(2)} \xi_3^{(1)}) + \ldots\ldots + \Pi_{1\,n-1} (\xi_{n-1}^{(2)} \xi_1^{(1)} - \xi_1^{(2)} \xi_{n-1}^{(1)}) +$$

$$+ \Pi_{23} (\xi_3^{(2)} \xi_2^{(1)} - \xi_2^{(2)} \xi_3^{(1)}) + \ldots\ldots + \Pi_{2\,n-1} (\xi_{n-1}^{(2)} \xi_2^{(1)} - \xi_2^{(2)} \xi_{n-1}^{(1)}) +$$

$$+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots +$$

$$+ \Pi_{n-2\,n-1} (\xi_{n-1}^{(2)} \xi_{n-2}^{(1)} - \xi_{n-2}^{(2)} \xi_{n-1}^{(1)}).$$

Forming, further, the analogous expressions

$$\sum_x \frac{\partial x_1}{\partial u_3} d \frac{\partial x_1}{\partial u_1}, \ldots\ldots, \sum_x \frac{\partial x_1}{\partial u_{n-1}} d \frac{\partial x_1}{\partial u_1},$$

$$\sum_x \frac{\partial x_1}{\partial u_3} d \frac{\partial x_1}{\partial u_2}, \ldots\ldots, \sum_x \frac{\partial x_1}{\partial u_{n-1}} d \frac{\partial x_1}{\partial u_2},$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\sum_x \frac{\partial x_1}{\partial u_{n-1}} d \frac{\partial x_1}{\partial u_{n-2}},$$

and solving these $\frac{(n-1)(n-2)}{2}$ equations, we find the rotations $p_{12}^{(i)}, \ldots\ldots,$ $p_{1\,n-1}^{(i)}$; $p_{23}^{(i)}, \ldots\ldots, p_{2\,n-1}^{(i)}$; etc., $p_{n-2\,n-1}^{(i)}$ (taking the coefficients of $du_1, du_2, \ldots\ldots,$ $du_{n-1}$ on the left-hand and the right-hand side):

$$
\left.\begin{aligned}
p_{12}^{(1)} &= \frac{1}{\Omega}\left|\begin{array}{llll}
-\frac{1}{2}\frac{\partial E_{11}}{\partial u_2}+\sum\limits_{\sigma=1}^{n-1}\xi_\sigma^{(1)}\frac{\partial \xi_\sigma^{(2)}}{\partial u_1} & \xi_3^{(2)}\xi_1^{(1)} & -\xi_1^{(2)}\xi_3^{(1)} & \cdots\\
-\frac{1}{2}\frac{\partial E_{11}}{\partial u_3}+\sum\limits_{\sigma=1}^{n-1}\xi_\sigma^{(1)}\frac{\partial \xi_\sigma^{(3)}}{\partial u_1} & \xi_3^{(3)}\xi_1^{(1)} & -\xi_1^{(3)}\xi_3^{(1)} & \cdots\\
\cdots\cdots\cdots & \cdots & \cdots & \cdots\\
-\frac{1}{2}\frac{\partial E_{11}}{\partial u_{n-1}}+\sum\limits_{\sigma=1}^{n-1}\xi_\sigma^{(1)}\frac{\partial \xi_\sigma^{(n-1)}}{\partial u_1} & \xi_3^{(n-1)}\xi_1^{(1)} & -\xi_1^{(n-1)}\xi_3^{(1)} & \cdots\\
\frac{1}{2}\left(\frac{\partial F_{23}}{\partial u_1}-\frac{\partial F_{12}}{\partial u_3}+\frac{\partial F_{13}}{\partial u_2}\right)-\sum\limits_{\sigma=1}^{n-1}\xi_\sigma^{(3)}\frac{\partial \xi_\sigma^{(2)}}{\partial u_1} & \xi_3^{(3)}\xi_1^{(2)} & -\xi_1^{(3)}\xi_3^{(2)} & \cdots\\
\cdots\cdots\cdots & \cdots & \cdots & \cdots\\
\cdots\cdots\cdots & \cdots & \cdots & \cdots\\
\frac{1}{2}\left(\frac{\partial F_{n-2\,n-1}}{\partial u_1}-\frac{\partial F_{1\,n-2}}{\partial u_{n-1}}+\frac{\partial F_{1\,n-1}}{\partial u_{n-2}}\right)-\sum\limits_{\sigma=1}^{n-1}\xi_\sigma^{(n-1)}\frac{\partial \xi_\sigma^{(n-2)}}{\partial u_1} & \xi_3^{(n-1)}\xi_1^{(n-2)} & -\xi_1^{(n-1)}\xi_3^{(n-2)} & \cdots
\end{array}\right|\\
&\qquad\text{etc., etc., etc.}\\
p_{13}^{(n-1)} &= \frac{1}{\Omega}\left|\begin{array}{llll}
\frac{1}{2}\left(\frac{\partial F_{2\,n-1}}{\partial u_1}+\frac{\partial F_{12}}{\partial u_{n-1}}-\frac{\partial F_{1\,n-1}}{\partial u_2}\right)-\sum\limits_{\sigma=1}^{n-1}\xi_\sigma^{(2)}\frac{\partial \xi_\sigma^{(1)}}{\partial u_{n-1}} & \xi_3^{(2)}\xi_1^{(1)} & -\xi_1^{(2)}\xi_3^{(1)} & \cdots\\
\frac{1}{2}\left(\frac{\partial F_{3\,n-1}}{\partial u_1}+\frac{\partial F_{13}}{\partial u_{n-1}}-\frac{\partial F_{1\,n-1}}{\partial u_3}\right)-\sum\limits_{\sigma=1}^{n-1}\xi_\sigma^{(3)}\frac{\partial \xi_\sigma^{(1)}}{\partial u_{n-1}} & \cdots\cdots & & \cdots\\
\cdots\cdots\cdots & \cdots & \cdots & \cdots\\
\frac{1}{2}\frac{\partial E_{n-1\,n-1}}{\partial u_1}-\sum\limits_{\sigma=1}^{n-1}\xi_\sigma^{(n-1)}\frac{\partial \xi_\sigma^{(1)}}{\partial u_{n-1}} & \cdots\cdots & & \cdots\\
\frac{1}{2}\left(\frac{\partial F_{3\,n-1}}{\partial u_2}+\frac{\partial F_{23}}{\partial u_{n-1}}-\frac{\partial F_{2\,n-1}}{\partial u_3}\right)-\sum\limits_{\sigma-1}^{n-1}\xi_\sigma^{(3)}\frac{\partial \xi_\sigma^{(2)}}{\partial u_{n-1}} & \cdots\cdots & & \cdots\\
\cdots\cdots\cdots & \cdots & \cdots & \cdots\\
\frac{1}{2}\frac{\partial E_{n-1\,n-1}}{\partial u_2}-\sum\limits_{\sigma-1}^{n-1}\xi_\sigma^{(n-1)}\frac{\partial \xi_\sigma^{(2)}}{\partial u_{n-1}} & \cdots\cdots & & \cdots\\
\cdots\cdots\cdots & \cdots & \cdots & \cdots\\
\frac{1}{2}\frac{\partial E_{n-1\,n-1}}{\partial u_{n-2}}-\sum\limits_{\sigma-1}^{n-1}\xi_\sigma^{(n-1)}\frac{\partial \xi_\sigma^{(n-2)}}{\partial u_{n-1}} & \cdots\cdots & & \cdots
\end{array}\right|\\
&\qquad\text{etc., etc., etc.}
\end{aligned}\right\}\quad(26)
$$

35. Let us finally consider the case, where the parametric lines

$u_1 = \text{const.},\ u_2 = \text{const.},\ \ldots\ldots,\ u_{n-2} = \text{const.};\ u_1 = \text{const.},\ \ldots\ldots,\ u_{n-3} = \text{const.},$
$u_{n-1} = \text{const.};\ \ldots\ldots\ldots;\ u_2 = \text{const.},\ \ldots\ldots,\ u_{n-1} = \text{const.}$

are the *lines of curvature* and the axes $x_{n-1}, \ldots, x_1$ coincide with their tangents. Then

$$\left.\begin{aligned} F_{12} = F_{13} = \ldots &= F_{1\,n-1} = \\ = F_{23} = \ldots &= F_{2\,n-1} = \\ = \ldots &= \\ &= F_{n-2\,n-1} = 0; \\ D_{12} = D_{13} = \ldots &= D_{1\,n-1} = \\ = D_{23} = \ldots &= D_{2\,n-1} = \\ = \ldots &= \\ &= D_{n-2\,n-1} = 0, \end{aligned}\right\} \tag{27}$$

and

$$\left.\begin{aligned} \xi_2^{(1)} &= \xi_3^{(1)} = \ldots = \xi_{n-1}^{(1)} = \\ = \xi_1^{(2)} &= \xi_3^{(2)} = \ldots = \xi_{n-1}^{(2)} = \\ = \ldots & \ldots\ldots\ldots\ldots\ldots = \\ = \xi_1^{(n-1)} &= \xi_2^{(n-1)} = \ldots = \xi_{n-2}^{(n-1)} = 0, \end{aligned}\right\} \tag{28}$$

$$\left.\begin{aligned} \xi_1^{(1)} &= \sqrt{E_{11}}, \\ \xi_2^{(2)} &= \sqrt{E_{22}}, \\ &\ldots\ldots\ldots \\ \xi_{n-1}^{(n-1)} &= \sqrt{E_{n-1\,n-1}}. \end{aligned}\right\} \tag{28'}$$

36. We get, further, from (25),

$$\left.\begin{aligned} p_{1n}^{(2)} &= p_{1n}^{(3)} = \ldots = p_{1n}^{(n-1)} = 0, \\ p_{2n}^{(1)} &= p_{2n}^{(3)} = \ldots = p_{2n}^{(n-1)} = 0, \\ p_{3n}^{(1)} &= p_{3n}^{(2)} = p_{3n}^{(4)} = \ldots = p_{3n}^{(n-1)} = 0, \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ p_{n-1n}^{(1)} &= p_{n-1n}^{(2)} = p_{n-1n}^{(3)} = \ldots = p_{n-1n}^{(n-2)} = 0, \end{aligned}\right\} \tag{29}$$

$(n-2)(n-1)$ of the first set of rotations are equal zero.

37. The other rotations of this set are given by the formulæ (25),

$$\left.\begin{aligned} p_{1n}^{(1)} &= D_{11}\xi_2^{(2)}\xi_3^{(3)}\ldots\ldots\xi_{n-1}^{(n-1)}\frac{(-1)^{n-1}}{D^2} = D_{11}\frac{1}{\sqrt{E_{11}}}\frac{(-1)^{n-1}}{D}, \\ p_{2n}^{(2)} &= D_{22}\xi_1^{(1)}\xi_3^{(3)}\ldots\ldots\xi_{n-1}^{(n)}\frac{(-1)^{n-1}}{D^2} = D_{22}\frac{1}{\sqrt{E_{22}}}\cdot\frac{(-1)^{n-1}}{D}, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ p_{n-1n}^{(n-1)} &= D_{n-1\,n-1}\xi_1^{(1)}\xi_2^{(2)}\ldots\xi_{n-2}^{(n-2)}\frac{(-1)^{n-1}}{D^2} = \\ &= D_{n-1\,n-1}\frac{1}{\sqrt{E_{n-1\,n-1}}}\cdot\frac{(-1)^{n-1}}{D}, \end{aligned}\right\} \tag{30}$$

38. Finally from (26),

$$
\left.
\begin{aligned}
p^{(1)}_{12} &= -\tfrac{1}{2}\frac{\partial E_{11}}{\partial u_2}\,\frac{1}{\xi^{(2)}_2\,\xi^{(1)}_1} = -\frac{1}{\xi^{(2)}_2}\,\frac{\partial \xi^{(1)}_1}{\partial u_2} = -\frac{1}{2\sqrt{E_{11}\,E_{22}}}\,\frac{\partial E_{11}}{\partial u_2},\\
p^{(1)}_{13} &= -\frac{1}{\xi^{(3)}_3}\,\frac{\partial \xi^{(1)}_1}{\partial u_3} = \frac{-1}{2\sqrt{E_{11}\,E_{33}}}\,\frac{\partial E_{11}}{\partial u_3},\\
&\cdots\cdots\cdots\cdots\cdots\cdots\\
p^{(1)}_{1\,n-1} &= -\frac{1}{\xi^{(n-1)}_{n-1}}\,\frac{\partial \xi^{(1)}_1}{\partial u_{n-1}} = \frac{-1}{2\sqrt{E_{11}\,E_{n-1\,n-1}}}\cdot\frac{\partial E_{11}}{\partial u_{n-1}},\\
p^{(1)}_{23} &= 0,\\
p^{(1)}_{24} &= 0,\\
&\cdots\cdots\\
p^{(1)}_{2\,n-1} &= 0,\\
&\cdots\cdots\\
p^{(1)}_{n-2\,n-1} &= 0.
\end{aligned}
\right\}\quad(31)
$$

$$
\left.
\begin{aligned}
p^{(2)}_{12} &= +\frac{1}{\xi^{(1)}_1}\,\frac{\partial \xi^{(2)}_2}{\partial u_1} = \frac{+1}{2\sqrt{E_{11}\,E_{22}}}\,\frac{\partial E_{22}}{\partial u_1},\\
p^{(2)}_{13} &= 0,\\
&\cdots\cdots\cdots\cdots\cdots\cdots\\
p^{(2)}_{1\,n-1} &= 0.\\
p^{(2)}_{23} &= -\frac{1}{\xi^{(3)}_3}\,\frac{\partial \xi^{(2)}_2}{\partial u_3} = \frac{-1}{2\sqrt{E_{22}\,E_{33}}}\cdot\frac{\partial E_{22}}{\partial u_3},\\
p^{(2)}_{24} &= -\frac{1}{\xi^{(4)}_4}\,\frac{\partial \xi^{(2)}_2}{\partial u_4} = \frac{-1}{2\sqrt{E_{22}\,E_{44}}}\cdot\frac{\partial E_{22}}{\partial u_4},\\
&\cdots\cdots\cdots\cdots\cdots\cdots\\
p^{(2)}_{2\,n-1} &= -\frac{1}{\xi^{(n-1)}_{n-1}}\,\frac{\partial \xi^{(2)}_2}{\partial u_{n-1}} = \frac{-1}{2\sqrt{E_{22}\,E_{n-1\,n-1}}}\,\frac{\partial E_{22}}{\partial u_{n-1}},\\
p^{(2)}_{34} &= 0,\\
&\cdots\cdots\\
p^{(2)}_{n-2\,n-1} &= 0.
\end{aligned}
\right\}\quad(32)
$$

$$
\left.
\begin{aligned}
p^{(3)}_{12} &= 0,\\
p^{(3)}_{13} &= +\frac{1}{\xi^{(1)}_1}\,\frac{\partial \xi^{(3)}_3}{\partial u_1} = \frac{+1}{2\sqrt{E_{11}\,E_{33}}}\,\frac{\partial E_{33}}{\partial u_1},\\
p^{(3)}_{14} &= 0,\\
&\cdots\cdots\\
p^{(3)}_{1\,n-1} &= 0,\\
p^{(3)}_{23} &= +\frac{1}{\xi^{(2)}_2}\,\frac{\partial \xi^{(3)}_3}{\partial u_2} = \frac{+1}{2\sqrt{E_{22}\,E_{33}}}\,\frac{\partial E_{33}}{\partial u_2},
\end{aligned}
\right\}\quad(33)
$$

$$\left.\begin{aligned}
p^{(3)}_{24} &= 0, \\
&\cdots\cdots \\
p^{(3)}_{2\,n-1} &= 0, \\
p^{(3)}_{34} &= \frac{-1}{\xi^{(4)}_4}\frac{\partial \xi^{(3)}_3}{\partial u_4} = \frac{-1}{2\sqrt{E_{33}E_{44}}}\frac{\partial E_{33}}{\partial u_4}, \\
p^{(3)}_{35} &= \frac{-1}{\xi^{(5)}_5}\frac{\partial \xi^{(3)}_3}{\partial u_5} = \frac{-1}{2\sqrt{E_{33}E_{55}}}\cdot\frac{\partial E_{33}}{\partial u_5}, \\
&\cdots\cdots\cdots\cdots\cdots\cdots \\
p^{(3)}_{3n-1} &= \frac{-1}{\xi^{(n-1)}_{n-1}}\cdot\frac{\partial \xi^{(3)}_3}{\partial u_{n-1}} = \frac{-1}{2\sqrt{E_{33}E_{n-1\,n-1}}}\frac{\partial E_{33}}{\partial u_{n-1}}, \\
p^{(3)}_{45} &= 0, \\
p^{(3)}_{46} &= 0, \\
&\cdots\cdots \\
p^{(3)}_{n-2\;n-1} &= 0.
\end{aligned}\right\} \text{(33) (Continued.)}$$

$$\left.\begin{aligned}
&\cdots\cdots \\
&\cdots\cdots \\
&\cdots\cdots
\end{aligned}\right\} (34)$$

$$\left.\begin{aligned}
p^{(n-1)}_{12} &= 0, \\
p^{(n-1)}_{13} &= 0, \\
&\cdots\cdots \\
p^{(n-1)}_{1\,n-2} &= 0, \\
p^{(n-1)}_{1\,n-1} &= \frac{+1}{\xi^{(1)}_1}\frac{\partial \xi^{(n-1)}_{n-1}}{\partial u_1} = \frac{+1}{2\sqrt{E_{11}E_{n-1\,n-1}}}\cdot\frac{\partial E_{n-1\,n-1}}{\partial u_1}, \\
p^{(n-1)}_{23} &= 0, \\
&\cdots\cdots \\
p^{(n-1)}_{2\,n-2} &= 0, \\
p^{(n-1)}_{2\,n-1} &= \frac{+1}{\xi^{(2)}_2}\frac{\partial \xi^{(n-1)}_{n-1}}{\partial u_2} = \frac{+1}{2\sqrt{E_{22}E_{n-1\,n-1}}}\frac{\partial E_{n-1\,n-1}}{\partial u_2}. \\
p^{(n-1)}_{34} &= 0, \\
&\cdots\cdots \\
p^{(n-1)}_{3\,n-2} &= 0, \\
p^{(n-1)}_{3\,n-1} &= \frac{+1}{\xi^{(3)}_3}\frac{\partial \xi^{(n-1)}_{n-1}}{\partial u_3} = \frac{+1}{2\sqrt{E_{33}E_{n-1\,n-1}}}\frac{\partial E_{n-1\,n-1}}{\partial u_3}, \\
&\cdots\cdots\cdots\cdots\cdots\cdots \\
&\cdots\cdots\cdots\cdots\cdots\cdots \\
p^{(n-1)}_{n-2\;n-1} &= \frac{+1}{\xi^{(n-2)}_{n-2}}\frac{\partial \xi^{(n-1)}_{n-1}}{\partial u_{n-2}} = \frac{+1}{2\sqrt{E_{n-2\,n-2}E_{n-1\,n-1}}}\frac{\partial E_{n-1\,n-1}}{\partial u_{n-2}};
\end{aligned}\right\} (35)$$

the $p$'s, which are not zero, are those which have two equal *indices*.*

---

*In the paper of Prof. Craig is an erratum: In the formula (66), (p. 155) the multipliers of the

39. It is now obvious that we may study the lines, the surfaces, etc., on the manifoldness of the $n-1$ dimensions (or that of $k$ generally) in the same manner as M. Darboux does.

GÖTTINGEN (Germany), *July*, 1899.

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brackets must be $\zeta''\zeta\zeta'$ and not $\xi''\xi$, $\xi'$, and, consequently, the formulæ of the page 156 must be written (with the $p_{13}$, $p'_{13}$, $p''_{13}$ of Professor Craig)

$$\begin{cases} p''_{12} = 0, & p'_{12} = \dfrac{+1}{2\sqrt{E_{11}\,E_{22}}} \cdot \dfrac{\partial E_{22}}{\partial t}, & p_{12} = \dfrac{-1}{2\sqrt{E_{11}\,E_{22}}} \, \dfrac{\partial E_{11}}{\partial u}, \\ p_{13} = \dfrac{+1}{2\sqrt{E_{33}\,E_{11}}} \cdot \dfrac{\partial E'_{11}}{\partial v}, & p'_{13} = 0, & p''_{13} = \dfrac{-1}{2\sqrt{E_{33}\,E_{11}}} \, \dfrac{\partial E_{33}}{\partial t}, \\ p''_{23} = \dfrac{+1}{2\sqrt{E_{22}\,E_{33}}} \, \dfrac{\partial E_{33}}{\partial u}, & p'_{23} = \dfrac{-1}{2\sqrt{E_{22}\,E_{33}}} \, \dfrac{\partial E_{22}}{\partial v}, & p_{23} = 0. \end{cases}$$